Zariski topology in $\mathbb{C}^n$

Reminder: a top. space $X$ is a set with a collection $\mathcal{A}$ of open & closed subsets such that:

- $U$ is open $\iff$ $X \setminus U$ closed
- $\emptyset, X$ open
- $\emptyset, X$ closed
- Any union of open subsets is open
- Any intersection of closed subsets is closed
- Finite intersection of open subsets is open
- Finite union of closed subsets is closed

Def (Zariski topology) $I$ = ideal in $\mathbb{C}[x_1, \ldots, x_n]$

$\sqrt{I} =$ zero set of $I$ (all polynomials in $I$ vanish)

$\sqrt{I}$ closed in Zariski topology.

This is a top. space.
Proof: \( \emptyset = \{ 1 = 0 \} \quad \mathbb{P}^n = V(0) \)

\[ V(\mathbb{C}[x_1, \ldots, x_n]) \]

- \( I_\mathbb{A} = \) any family of ideals

\[ \bigcap_{\alpha} V(I_\alpha) = V(\sum_{\alpha} I_\alpha) \]

\[ \sum_{\alpha} I_\alpha \ell = \sum_{\ell} I_{\alpha_\ell} \]

\( f_{\alpha_\ell} \in I_{\alpha_\ell} \)

\( f_{\alpha_\ell} \) vanishes on \( V(I_{\alpha_\ell}) \)

\( \Rightarrow \) vanishes on \( \bigcap_{\alpha} V(I_\alpha) \)

\( \Rightarrow \) all \( f_{\alpha_\ell} \) vanish on \( \bigcap_{\alpha} V(I_\alpha) \)

\( \Rightarrow f_{\alpha_1} \ldots + f_{\alpha_k} \) also vanish

Conversely, if \( p \in V(\sum_{\alpha} I_\alpha) \)

all functions from \( \sum_{\alpha} I_\alpha \)
vanish at \( p \)

\( \Rightarrow \) all functions from \( I_\alpha \) vanish

\( \Rightarrow p \in V(I_\alpha) \) for all \( \alpha \)
\[ \Rightarrow p \in V(I_d) \text{ for all } d, \text{ at } p \]

\[ V(I_1) \cup V(I_2) = V(I_1, I_2) \text{ } \text{Exercise} \]

**Very weird topology:**

For all ideals \( I \)

\[ V(f) = \begin{cases} \emptyset & f = 0 \\ \{ x \in X \mid f(x) = \text{const} \} & f \neq 0 \end{cases} \]

In any point, \( \deg f = n \)

All closed subsets = either

- \( X \)
- or finite subset

Any finite subset is closed

(Find a polynomial with given roots).

Open = complements to finite subsets.

(= empty or dense in \( X \)

in "usual" topology).

If \( f \neq 0 \)

**Remark:** Not Hausdorff, no disjoint

open sets; thus the topology.
open subsets. Any two nonempty open subsets intersect.

Ex: In $\mathbb{C}^2$, $\{x^2 = y^3\}$ closed

$\{x^2 \neq y^3\}$ open

In $\mathbb{C}^2$, complement to $\{x^2 = y^3\}$ is open

$(0,0) = V(x, y) = \{x = 0\} \cup \{y = 0\}$

Complement to $(0,0) = \{x + y \neq 0\}$

Fact: Zariski closed subset in $\mathbb{C}^n$ \Rightarrow closed in "usual" topology

Zariski open subset in $\mathbb{C}^n$ \Rightarrow open and dense in "usual" topology (or empty).

Caution: Zariski topology in $\mathbb{C}^2$ is not the product topology.
It is NOT the product topology. 

**Claim:** \( f: \mathbb{C}^n \rightarrow \mathbb{C}^m \) algebraic function

\[
f: (a_1, \ldots, a_n) \mapsto (\varphi_1(a_1, \ldots, a_n), \ldots, \varphi_m(a_1, \ldots, a_n))
\]

\( \varphi_i = \text{some polynomials in } n \text{ variables} \)

\( f^*: \mathbb{C}[y_1, \ldots, y_m] \rightarrow \mathbb{C}[x_1, \ldots, x_n] \) (pullback) homomorphism

**Claim:** \( f \) is continuous in Zariski topology.

**Recall:** \( f \) is continuous \( \Rightarrow \) \( f^{-1}(\text{open subset}) = \text{open} \) \( \Rightarrow \) \( f^{-1}(\text{closed subset}) = \text{closed} \).

\[
f: \mathbb{C}^n \rightarrow \mathbb{C}^m
\]

closed subset of \( \mathbb{C}^n \) = \( V(I) \)

\( I = \text{ideal in } \mathbb{C}[y_1, \ldots, y_m] \)

\[
f^{-1}(V(I)) = \{ p \in \mathbb{C}^n : f(p) \in V(I) \}
\]

= \{ p : \text{all polynomials } g \in I \}
\[ \begin{align*}
\{ p : \text{all polynomials } g \in I \text{ vanish at } f(p) \} \\
= \{ p : g(p_1, \ldots, p_n), g_2(p_1, \ldots, p_n), \ldots, g_m(p_1, \ldots, p_n) = 0 \} \\
\Rightarrow \\
\{ p : f^*(g) \text{ vanishes at } p \} \\
\text{Conclusion: } f^{-1}(V(I)) = V(f^{-1}(I)) \\
\Rightarrow f^{-1}(\text{closed}) \text{ is closed.}
\end{align*} \]

Zariski topology for Spec A

\begin{align*}
A & \text{ ring} \\
\text{Spec } A & = \{ \text{prime ideals in } A \} \\
I & = \text{ideal in } A \\
V(I) & = \{ \text{prime ideals } p \supset I \} \\
\text{Zariski topology: } V(I) & \text{ are closed}
\end{align*}

Note: \( A = \mathbb{C}[x_1, \ldots, x_n] \)

\[ p = (x_1 - a_1, \ldots, x_n - a_n) \text{ is maximal ideal} \]

\[ I \subset p \iff I \subset (x_1 - a_1, \ldots, x_n - a_n) \]
\[ I \subseteq \text{Spec} A \iff I \subseteq \langle x_1 - a_1, \ldots, x_n - a_n \rangle \]

Thus, all functions in \( I \) vanish at \((a_1, \ldots, a_n)\).

(a) \( \text{Spec} A \) is a topological space with Zariski topology.

(b) \( f : A \to B \) is a ring homomorphism.

\( f : \text{Spec} B \to \text{Spec} A \) continuous in Zariski topology.

\[ \text{Proof:} \quad (a) \quad \emptyset = \bigcap \text{Spec} A \]

\[ \text{Spec} A = \bigcap \{ (a) \} \]

\[ P \supseteq A \iff P = A \]

but \( P \) proper

Arbitrary intersections

\[ \bigcap_{a} \text{Spec} (I_a) = \{ P \supseteq \sum_{a} I_a \} \]

Finite unions

\[ \text{Spec} (I_1 \cup I_2) = \text{Spec} (I_1, I_2) \]

\[ \{ P \supseteq I_1 \} \text{ or } \{ P \supseteq I_2 \} \]
$I_2 = \text{ideal generated by } fg, f \in I_1, g \in I_2$

- If $p = \mathfrak{p}$, then $f \in \mathfrak{p}$ since $p$ is an ideal, $fg \in \mathfrak{p}$ \(\Rightarrow\) all such $fg \in \mathfrak{p}$
- If $fg \in \mathfrak{p}$, since $\mathfrak{p}$ is prime, either $f$ or $g \in \mathfrak{p}$ \(\Rightarrow\) $I_1 \subset \mathfrak{p}$

Allows to define topology on $\text{Spec } \mathcal{O}_X$, $\text{Spec } K[x]$

$k = \text{any field}$

(6) $f^* : A \rightarrow B$

$f : \text{Spec } B \rightarrow \text{Spec } A$

\[ f^*(V(I)) = \{ \text{prime ideals } \mathfrak{p} \in B | f(\mathfrak{p}) \supset I \} \]

\[ f^*(\mathfrak{p}) \supset f^*(I) \Leftrightarrow f^*(p) \supset I \]

Claim: $f^*(V(I)) = V(f^*(I))$
Ex. \( A = \mathbb{C}[x_1, \ldots, x_n] \) 

\[ \text{Spec } A \supset \text{ maximal ideals } = \text{ maximal ideals in } \mathbb{C}[x_1, \ldots, x_n] \text{ containing } \mathcal{J} \]

\[ = \overline{V(\mathcal{J})} = \text{zero set of } \mathcal{J}. \]

\[ \mathcal{I} = \text{ideal in } \mathbb{C}[x_1, \ldots, x_n] \rightarrow \mathcal{I} = \text{ideal in } \mathbb{C}[x_1, \ldots, x_n] \text{ containing } \mathcal{J}. \]

\[ V(\mathcal{I}) = V(\bar{\mathcal{I}}) \subseteq V(\mathcal{J}) \]

Zariski closed subset of \( V(\mathcal{J}) \)

= intersection of closed subsets in \( \mathbb{C}^n \) with \( V(\mathcal{J}) \)

= closed subset in \( \mathbb{C}^n \) contained in \( V(\mathcal{J}) \).

Ex. \( \mathcal{J} = \langle x^2 - y^3 \rangle \), \( V(\mathcal{J}) \) is contained in \( \mathbb{C}^2 \).

Zariski closed subset of \( \mathbb{C}^2 \); \( \mathbb{C}^2 \), \( V(x^2 - y^3) \).
Zariski closed subset $D(x^2 - y^3)$

$\cap I$ for ideal

$I$ contains $x^2 - y^3$.

- \text{Hom: Spec } \mathbb{K}[x, x^4]