The Question: Can sutured annular Khovanov homology be used to solve the word/conjugacy problem in $B_n$?

Prelims:

(Def): $n$-strand Braid Group:

$B_n := \langle \sigma_1, \ldots, \sigma_{n-1} | \sigma_i \sigma_j \sigma_i = \sigma_j \sigma_i \sigma_j \ (|i-j| \geq 2) \rangle$

... where each $\sigma_i$ is the Artin generator:

\[
\sigma_i = \begin{array}{c}
\includegraphics{braid_diagram.png}
\end{array}
\]

- Word Problem: Let $w \ & w'$ be words in the generators of $B_n$, & let $\sigma(w)$ denote the braid of $w$, then is $\sigma(w) = \sigma(w')$?

- Conjugacy Problem: $\sigma(w) \sim \sigma(w')$?
Answer:

word problem? Yes!

conjugacy problem? Not Quite!

Recall that Sutured Annular Khovanov homology associates to an oriented link \( L \subset A \times I \) (\( A \) an annulus) a \textit{tri-graded} vector space:

\[
SKh(L) = \bigoplus_{i,j,k} SKh^i(L; j, k)
\]

\( SKh(L) \) is an \textit{invariant} of oriented isotopy classes of links \( L \subset A \times I \). It is constructed as follows:

\( A \times I \) is an \textit{annular braid closure}\( \widehat{\tau} \) in \( A \times I \) is:

\[\text{... then we project } \widehat{\tau} \text{ onto the } A \times S^1 \\text{ of } A \times I, \text{ like:}\]
... then we project \( \mathcal{B} \) onto the \( \mathbb{R} \times 2/3 \) "slice" of \( \mathbb{A} \times \mathbb{I} \), like:

\[
\mathbb{A} \times \mathbb{R}^{1/2}  \]

**Note:** Conjugate braids have isotopic closures in \( \mathbb{A} \times \mathbb{I} \).

... then we close up the inner and outer circles of \( \mathbb{A} \times \mathbb{R}^{1/2} \), so we have our diagram \( D \) in \( \mathbb{S}^2 \setminus \{ N, S \} \):

\[
\begin{array}{ccc}
\bigcirc & \rightarrow & \bigcirc \bigcirc \\
\end{array}
\]

... if we forget about \( N \), we have a diagram \( D \) in \( \mathbb{S}^2 \setminus \{ 0, \infty \} \times \mathbb{R}^2 \) and we can form the ordinary Kleinian complex of \( D \), denoted \( CK\mathcal{B} \), ...

... to get \( S\mathcal{K}\mathcal{B} \), choose an arc:

\[
\begin{array}{c}
\bigcirc \bigcirc \\
S
\end{array}
\]

\[
\begin{array}{c}
\bigcirc \bigcirc \\
N
\end{array}
\]

\[
\bigcirc \bigcirc
\]

s.t. \( \gamma \) passes all crossings of \( \mathcal{B} \)

(Def): \( k \)-grading: algebraic intersection \( k \) of the oriented resolutions of \( \mathcal{B} \) generating \( CK\mathcal{B} \) and \( \mathcal{B} \)
**Lemma:** 1 doesn't increase k-grading. (Roberts)

\[ \Rightarrow \text{The } k\text{-grading gives us a filtration:} \]

\[ 0 \subset \ldots \subset F_{n-1}(D) \subset F_n(D) \subset \ldots \subset Ckh(D) \]

\[ \ldots F_n(D) : \text{ subcomplex of } Ckh(D) \text{ bounded by resolutions } \]
\[ w/ \text{ k-grading } \leq n. \]

**(Def):** \( F_n(D; i) = F_n(D) \cap \bigoplus_i Ckh^i(D; i) \)

\[ \Rightarrow \text{subured annular Khovanov homology groups:} \]

\[ S\text{kh}^i(L; j, k) = H^i \left( \frac{F_k(D; i)}{F_{k-1}(D; i)} \right) \]
Thm: \( \text{Sym}(S) \cong \text{Sym} (\mathbb{A}) \) implies \( \sigma \cdot \mathbb{A} \).

So word problem solved, but what about the conjugacy problem?

(Def): Reversal braid: For \( \sigma \in \mathbb{B}_n \), where
\[ \sigma = \sigma (w^s) \] then the reversal of \( \sigma \), denoted \( \sigma^r \), is:
\[ \sigma^r = \sigma (w^s) \]

(Def): Flype: Let \( \sigma \) & \( \sigma^r \) be 3-braids, then a flype is the following transformation:

\[ \text{Diagram showing flype transformation} \]

Note that, given projections \( D \) & \( D^r \) of closures \( \sigma \) & \( \sigma^r \) respectively, they look like:
Thm: Let \( \mathcal{D} \in \text{Br} \), then \( \text{Skh}(\mathcal{D}) \cong \text{Skh}(\mathcal{D}') \).

Proof. Given \( \mathcal{D} \) & \( \mathcal{D}' \) and respective projections \( \mathcal{D} \) and \( \mathcal{D}' \) onto \( A \times \mathbb{E}^2 \), there is a bijection correspondence between the oriented resolutions of \( \mathcal{D} \) & \( \mathcal{D}' \), so \( \text{Skh}(\mathcal{D}) \cong \text{Skh}(\mathcal{D}') \).

Corollary: There are infinitely many pairs \( \left( \mathcal{D}, \mathcal{D}' \right) \) s.t. \( \mathcal{D} \neq \mathcal{D}' \) but \( \text{Skh}(\mathcal{D}) \cong \text{Skh}(\mathcal{D}') \).

Proof. Suppose \( \mathcal{D} \) & \( \mathcal{D}' \) are braids related by a flype, then \( \mathcal{D}' \) is isotopic to \( \mathcal{D} \) in \( \mathcal{A} \times \mathbb{I} \), so \( \mathcal{D}' \) is a transverse mirror to \( \mathcal{D} \). Thus, by theorem, \( \text{Skh}(\mathcal{D}) \cong \text{Skh}(\mathcal{D}') \).

... (Birman & Menasco) There are infinitely many distinct pairs of braids related by a flype s.t. the flype changes the conjugacy class.

... but maybe?

Open Questions:
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(i) Does \( \text{Skh}(\hat{\sigma}) \cong \text{Skh}(\hat{\sigma}') \turnstile \hat{\sigma} \cong \hat{\sigma}' \) or \( \hat{\sigma} \cong \hat{\sigma}' \)?

(ii) What if we also assume \( \hat{\sigma} & \hat{\sigma}' \) are alternating braids?

(iii) What if \( \text{Skh}(\hat{\sigma}^k) \cong \text{Skh}(\hat{\sigma}'^k) \forall k \geq 0 \)?

... (ii) & (iii) combined? Link Floer homology is promising here, but? for Skh.

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Sketch of main theorem proof.

Plamenevskaya's Invariant: \( \tilde{\psi}(L) \)

(Def): \( \tilde{\psi}(L) \): Given link \( L \) with diagram, given by braid closure \( \hat{\sigma} \), take the oriented resolution obtained by taking the \( 0 \)-resolution of positive crossings & the \( 1 \)-resolution of negative crossings. Let \( \tilde{\psi}(L) = u_1 \circ u_2 \circ ... \circ u_n \in \mathbb{V} \).

Lemma: \( \tilde{\psi}(L) \) is a cycle.

Proof: \( \rho \) on \( \text{CKh}(L) \) is the sum of maps for all edges \( \sigma \) with \( \rho \) as their initial end. Since \( \tilde{\psi}(L) \in \mathbb{V} \), \( \text{CKh}(L) \), and when \( 0 \)-res of \( + \)-crossing becomes \( - \)-res, the two circles merge.
\[ O \to \mathcal{C} \]

\[ \Rightarrow d_{\mathcal{C}} \text{ is multiplication (recall: } m(u, \mathcal{C}u) = 0) \]
\[ \Rightarrow d_{\mathcal{C}}(\mathcal{C}(u)) = 0, \text{ so } d(\mathcal{C}(D)) = 0. \]

**Def:** \[ \mathcal{C}(u) \text{ is the homology class of } \mathcal{C}(u). \]

**Thm:** \[ \mathcal{C}(u) \text{ is an invariant of the } \mathcal{C}\text{-equivariant links } L \in \mathcal{C}(R^{3}, S_{4}(k)) \text{ up to sign.} \]

**Def:** \[ D_{n} : \quad \text{(unit disk with } n \text{ points on } R \text{-axis)} \]

**Def:** \[ \text{Admissible arc: } \gamma : I \to D_{n} \text{ s.t.} \]
\[ \gamma \text{ is a smooth embedding, } \gamma \text{ transverse to } \partial D, \text{ satisfying } \gamma(0) = -1 \in \partial D, \gamma(1) = 0, \text{ and } \gamma(t) \in \text{Int}(D_{n}) \text{ for } t \neq 0, 1, \text{ and } \frac{d\gamma}{dt} \neq 0 \text{ for all } t. \]

\[ \text{So like: } \]

\[ \text{Qz an admissible arc} \]

**Def:** \[ \text{Pulled Tight: } \gamma \text{ and } \gamma' \text{ are pulled.} \]
(Def): Pulled Tight: \( \gamma \) and \( \gamma' \) are pulled tight if \( \gamma \cap \gamma' \) or \( \gamma \) intersects \( \gamma' \) transversally there are no empty bigons, like:

\[
\begin{align*}
\text{...} & x_1, x_2, \ldots, x_z \in \mathbb{R}^2 \text{ then } \gamma([x_1, x_2]) \cup x'(\mathbb{C}) \text{ bounds a disk } A \subset \mathbb{C}, \text{ at least one } y \in A.
\end{align*}
\]

Lemma: Up to isotopy, we can always "pull tight" two admissible arcs.

(Def): Right: \( \gamma \) is right of \( \gamma' \) if, after they are pulled tight, orientation induced by tangent vectors \( \frac{\partial \gamma}{\partial \theta} \big|_{\theta = 0} \) and \( \frac{\partial \gamma'}{\partial \theta} \big|_{\theta = 0} \) agrees with the standard orientation on \( \mathbb{C} \).

\[
\begin{align*}
\text{...} & y \text{ starts above } \gamma \text{ if } \gamma \text{ right of } \gamma'.
\end{align*}
\]

Thm: \( B_n \cong \text{MCG}(D_n) \)

... identify each \( \gamma_i \) with \( D_n \rightarrow D_n \) defined by:
\( i \circ id \text{ on } D_{n} \text{ except for a small disc containing } \\
\partial_{2} \text{ on this disc its a counterclock.} \)

\[ \Rightarrow \text{ braids act on } D_{n} \text{ from the right, so } B_{n} \]
\( \text{acts on admissible arcs up to isotopy. (8)} \)
\( \text{or merely convention, could choose negative Artin generators.} \)

\[ \text{Def: Right Veering: } \sigma \in B_{n} \text{ right-veering} \]
if \( \forall \gamma, (\gamma) \sigma \text{ is right of } \gamma \text{ when pulled tight. (Left veering same deal)!} \)

\[ \text{Lemma: If } \sigma \in B_{n} \text{ is left & right veering, then } \sigma = 1. \]

...follows from Alexander Lemma & fact that \( \sigma \)
(\( \sigma \text{ isotopic to a map that fixes all "nice" admissible arcs.} \)

\[ \text{Then: if } \sigma \text{ isn't right-veering, then } \psi(\sigma^{-1}) = 0. \)

\[ \text{Corollary: If } \psi(\sigma^{-1}) \neq 0 \text{ and } \psi(\sigma^{-1}) \neq 0, \]

\textbf{Corollary}: If } \psi(\mathcal{O}) \neq 0 \text{ and } \psi(m(\mathcal{O})) \neq 0, \text{ then } \mathfrak{g} = \mathfrak{h}.

... filtration of } \text{Ch} \text{ by } k\text{-grading yields a spectral sequence r. t. : } E_1 = \text{Sh}(L), \text{ } E_{\infty} = \text{K}(L) \hfill \\
& \& \& \text{on shifts the triple grading } (n, j, k) \text{ by } (1, 0, -n). \hfill \\
\textbf{Proof. (of 1):} \text{ Suppose } \text{Skh}(\mathcal{O}) \cong \text{Sh}(L). \hfill \\
\textbf{(Roberts):} \psi(\mathcal{O}) \text{ is the image of the bottom } k\text{-grading of } \text{Skh}(\mathcal{O}) \text{ under the spectral sequence mentioned above.} \hfill \\
\text{A computation of the spectral sequence shows } \text{Skh}(\mathcal{O}) \text{ collapses immediately and } \psi(\mathcal{O}) \text{ survives.} \hfill \\
\textbf{Then (Roberts):} \text{ Skh symmetric under taking mirrors; } \text{Skh}_{s,n,k}(L) \cong \text{Sh}_{s,n+k}(m(L)), \text{ and the spectral sequence converging to } \text{K}(L) \text{ is filtered chain isomorphic to that induced on } \text{Skh}_{s,n,k}(m(L)) \text{ by higher differentials on } \text{Skh}_{s,n,k}(m(L)). \hfill \\
\text{... symmetry implies the spectral sequence } \text{Skh}(m(\mathcal{O})) \text{ to } \text{K}(m(\mathcal{O})) \text{ collapses immediately and } \psi(m(\mathcal{O})) \neq 0 \hfill \\
\[ \Rightarrow \sigma = 1. \]

**Note:** Roberts defines "Khomov Stein homology," which is an equivalent construction.

**Remark:** All results hold for HFL.

**Remark:** Speed of Skh?

(Garside - Thurston): \( \exists \) an algorithmic solution to the word problem in \( B_n \) with complexity \( O(\sqrt{\nu n \log n}) \).

... further improved to \( O(\nu^n) \) (Birm - Lee)

Sources:

(Roberts): L. P. Roberts. "On Knot Floer homology in double branched covers"

(J. A. Baldwin & E. Grigsby): "Categorified invariants & the braid group"

(O. Plamenevskaya): "Transverse knots & Khovanov homology."