Open Book Decompositions (Sharon Rubin)

From:

Lectures on open book decompositions and contact structures.

— John Etnyre

— M is always a closed, oriented, 3-manifold.

Main Theorem (Giroux 2000):

$$\left\{\begin{matrix}\text{Oriented contact}\\ \text{structures on } M\end{matrix}\right\} \Big/ \text{isotopy} \longleftrightarrow \left\{\begin{matrix}\text{open book}\\ \text{decompositions on } M\end{matrix}\right\} \Big/ \begin{matrix}\text{positive}\\ \text{stabilization}\end{matrix}$$

Def^n: An open book decomposition of M is a pair (B, π), with

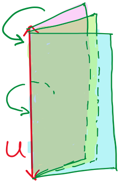
- B an oriented link in M, called the binding
- $\pi : M \setminus B \longrightarrow S^1$ a fibration w/ $\pi^{-1}(\theta)$ the interior of a compact surface $\Sigma_\theta \subset M$ and $\partial \Sigma_\theta = B$ $\forall \theta \in S^1$.

$\Sigma = \Sigma_\theta$ is called the page.

Examples: (1): Take $S^3 \subset \mathbb{C}^2_{(z_1, z_2)}$, and the unknot $U = \{z_1 = 0\} \subset S^3$. The complement fibers as

$$\pi_U : S^3 \setminus U \longrightarrow S^1$$
$$(z_1, z_2) \longmapsto \frac{z_1}{|z_1|}$$

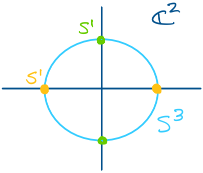
(or $\pi_U(r_1 e^{i\theta_1}, r_2 e^{i\theta_2}) = \theta_1$).



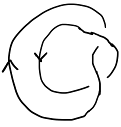
In $\mathbb{R}^3 \subset S^3$, U is the z-axis, and the pages are closed half planes of constant θ.

(2): Again $S^3 \subset \mathbb{C}^2$. Let $H^+ = \{z_1 z_2 = 0\} \subset S^3$
$H^- = \{z_1 \overline{z_2} = 0\} \subset S^3$.

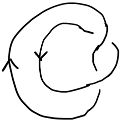
They have different orientations



H^+ (H^-) is the positive (negative) Hopf link.



H^+



H^-

The fibrations are

$$\pi_+ : S^3 \setminus H^+ \longrightarrow S^1 \qquad \pi_- : S^3 \setminus H^- \longrightarrow S^1$$
$$(z_1, z_2) \longmapsto \frac{z_1 z_2}{|z_1 z_2|} \qquad (z_1, z_2) \longmapsto \frac{z_1 \overline{z_2}}{|z_1 \overline{z_2}|}$$

(or $\pi_\pm(r_1 e^{i\theta_1}, r_2 e^{i\theta_2}) = \theta_1 \pm \theta_2$)

The pages are annuli:

(Henry Segerman)

Def^n: An abstract open book is a pair (Σ, ϕ) with

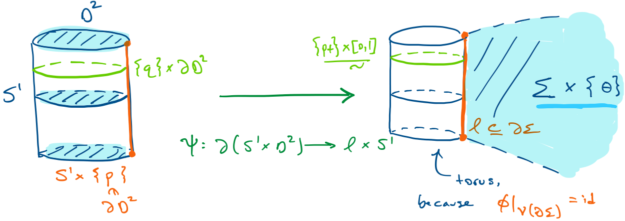
- Σ an oriented compact surface with boundary
- A diffeomorphism $\phi : \Sigma \longrightarrow \Sigma$ called the monodromy with $\phi|_{\nu(\partial\Sigma)} = \mathrm{id}$.

An equivalence of open books (Σ_1, ϕ_1) and (Σ_2, ϕ_2) is a diffeomorphism $h : \Sigma_1 \longrightarrow \Sigma_2$ satisfying

$$h \circ \phi_2 = \phi_1 \circ h.$$

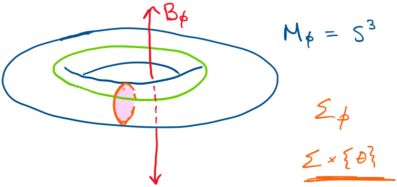
- An abstract open book (Σ, ϕ) yields a closed, oriented 3-manifold

$$M_\phi := \underbrace{\frac{\Sigma \times [0,1]}{(\phi(x), 0) \sim (x, 1)}}_{\text{Mapping torus} =: \Sigma_\phi} \cup_\psi \left(\bigsqcup_{\{\text{components of } \partial\Sigma\}} S^1 \times D^2 \right)$$

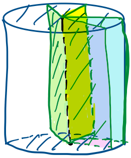

D^2
S^1
$\{q\} \times \partial D^2$
$S^1 \times \{p\}$
$\overset{\cap}{\partial D^2}$
$\{pt\} \times [0,1] / \sim$
$\psi : \partial(S^1 \times D^2) \longrightarrow \ell \times S^1$
$\Sigma \times \{\theta\}$
$\ell \subseteq \partial\Sigma$
torus, because $\phi|_{\nu(\partial\Sigma)} = \mathrm{id}$

Denote the union of the cores of the solid tori by B_ϕ

Example: $(\Sigma, \phi) = (D^2, \mathrm{id} : D^2 \longrightarrow D^2)$. Mapping torus is a solid torus



- (Σ, ϕ) to (B, π): A standard $S^1 \times D^2$ has "pages"



In M_ϕ, each of these connects via ψ to a $\Sigma \times \{\theta\} \subseteq \Sigma_\phi$. The union of these two is a page in M_ϕ.

- In our example, (D^2, id) produces Example (i), (U, π_U), an OBD on S^3.

- In general, (Σ_g^n, id) produces the 3-manifold $\#_{2g+n-1} S^1 \times S^2$.

Lemma: (i): an OBD (B, π) on M gives an abstract open book (Σ_π, ϕ_π) with
$\hookrightarrow$ monodromy around base S^1 of π

$(M_{\phi_\pi}, B_{\phi_\pi})$ diffeomorphic to (B, π)

(ii): (Σ, ϕ) gives an OBD (B_ϕ, π_ϕ) on M_ϕ up to diffeomorphism

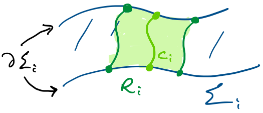
(iii) Equivalent abstract open books give diffeomorphic 3-manifolds.

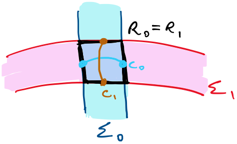
Remark:
- We sketched the proof of (ii) above
- (B, π) is finer than (Σ, ϕ), because the former can be considered up to isotopy in M.

Theorem (Alexander 1920): Every closed oriented 3-manifold has an OBD.

Def^n: Let (Σ_i, ϕ_i), $i = 0, 1$ be abstract open books, and $c_i \subseteq \Sigma_i$ properly embedded arcs $(\partial c_i \subseteq \partial \Sigma_i)$. Form the surface

$$\Sigma_0 * \Sigma_1 = \Sigma_0 \cup_{R_0 = R_1} \Sigma_1$$



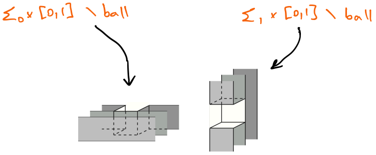


The Murasugi sum is the abstract open book

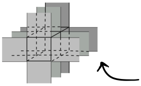
$$(\Sigma_0, \phi_0) * (\Sigma_1, \phi_1) := (\Sigma_0 * \Sigma_1, \phi_0 \circ \phi_1)$$

Theorem (Gabai, 1983): $M_{(\Sigma_0, \phi_0)} \# M_{(\Sigma_1, \phi_1)}$ is diffeomorphic to $M_{(\Sigma_0, \phi_0) * (\Sigma_1, \phi_1)}$.

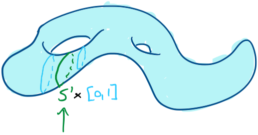
Sketch of Proof:

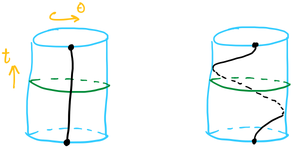


$\Sigma_0 \times [0,1] \setminus$ ball
$\Sigma_1 \times [0,1] \setminus$ ball



Dehn twist

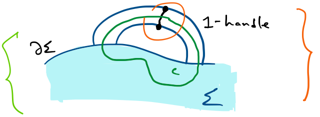




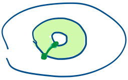
$$(t, \theta) \longmapsto (\theta + 2\pi t, t)$$

Def^n: A **± - stabilization** of an abstract open book (Σ, ϕ) is the open book $S_\pm(\Sigma, \phi) := (\Sigma', \phi')$, with

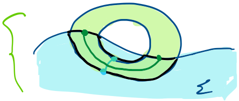
- $\Sigma' = \Sigma \cup$ 1-handle
- $\phi' = \phi \circ T_c$, where T_c is a right / left (+/−)-Dehn twist along a curve $c \subseteq \Sigma'$ which intersects the co-core of the 1-handle exactly once



Note: Pages of $(H^\pm, \pi_\pm)$ are annuli,



So the page of $(\Sigma, \phi) * (H^\pm, \pi_\pm)$ looks like



$$= \underline{\Sigma'}.$$

Hence:

$$\boxed{\begin{aligned} &S_\pm(\Sigma, \phi) = (\Sigma, \phi) * (H^\pm, \pi_\pm), \\ \text{and} \quad &M_{S_\pm(\Sigma,\phi)} = M_{(\Sigma,\phi)} \# \overbrace{M_{(H^\pm, \pi_\pm)}}^{S^3} = M_{(\Sigma,\phi)}. \end{aligned}}$$

Def^n: A contact structure ξ on M is supported by the open book decomposition (B, π) if ξ is isotopic to $\ker(\alpha)$, where α is a 1-form satisfying

- $\alpha > 0$ on B
- $d\alpha > 0$ on each Σ_θ

Lemma: (M, ξ) is supported by (B, π) if and only if, on compact subsets of the Σ_θ, ξ can be isotoped arbitrarily close to $T\Sigma_\theta$, and in such a way that (at least eventually in the isotopy) $\xi_t \pitchfork B$ and $\xi_t \pitchfork (\Sigma_\theta \cap \underbrace{\nu(B)}_{\text{fixed}})$

Example: (S^3, ξ_{std}) is supported by (U, π_U) (picture the Lemma).

$$\xi_{std} = \xi_{rot} = \ker(\underbrace{r_1^2\, d\theta_1 + r_2^2\, d\theta_2}_{\alpha_{rot}})$$

$$\pi_U : S^3 \setminus U \longrightarrow S^1$$
$$(r_1, \theta_1, r_2, \theta_2) \longmapsto \theta_1$$

For fixed θ_0, the page $\pi_U^{-1}(\theta_0)$ is parameterized by $f(r, \theta) = (\sqrt{1 - r^2}, \theta_0, r, \theta)$.

$$d(f^*(\alpha_{rot})) = 2r\, dr \wedge d\theta > 0$$
$$\alpha_{rot}(TU) = \alpha_{rot}(\partial_{\theta_2}) = r_2^2 > 0$$

Remark: Similar calculations show that (H^+, π_+) supports (S^3, ξ_{std}), but that (H^-, π_-) does not. $\left(f(r,\theta) = (\sqrt{1-r^2}, \theta_0 - \theta, r, \pm\theta) \right)$

Theorem (Thurston-Winkelnkemper 1975): Every open book decomposition supports a contact structure...

Theorem (Giroux 2000): ... uniquely, up to isotopy.

- Denote by $\xi_{(\Sigma,\phi)}$ the contact structure supported by (Σ,ϕ). Now

$$\boxed{\xi_{(\Sigma_0,\phi_0)} \# \xi_{(\Sigma_1,\phi_1)} = \xi_{(\Sigma_0,\phi_0)*(\Sigma_1,\phi_1)}},$$

with proof the same picture as before.

Corollary (Giroux 2000): $M_{S_\pm(\Sigma,\phi)}$ is diffeomorphic to $M_{(\Sigma,\phi)}$ and $\xi_{S_+(\Sigma,\phi)}$ is isotopic to $\xi_{(\Sigma,\phi)}$.

Proof: The first was shown earlier. The second is the same equation with more justification.

$$\begin{aligned}
\xi_{S_+(\Sigma,\phi)} &= \xi_{(\Sigma,\phi)*(H^+,\pi_+)} \\
&= \xi_{(\Sigma,\phi)} \# \xi_{(H^+,\pi_+)} \\
&= \xi_{(\Sigma,\phi)} \# \xi_{std}, S^3 \quad \text{(The OBD } (H^+,\pi_+) \text{ supports } (S^3,\xi_{std})\text{)} \\
{*} &= \xi_{(\Sigma,\phi)}
\end{aligned}$$

* To form a contact connected sum, you remove Darboux balls from each summand and glue. But S^3 is the union of two Darboux balls, so removing one and gluing the complement to $(M_\phi, \xi_{(\Sigma,\phi)}) \setminus$ Darboux ball just gives $(M_\phi, \xi_{(\Sigma,\phi)})$. ■

Remark: The OBD (H^-,π_-) does not support (S^3, ξ_{std}), and

$\xi_{S_-(\Sigma,\phi)}$ is not isotopic to $\xi_{(\Sigma,\phi)}$. They are not even homotopic.

We now have a well-defined map:

$$\psi : \left\{ \begin{matrix} \text{Oriented contact} \\ \text{structures on } M \end{matrix} \right\} \Big/ \text{isotopy} \longleftarrow \left\{ \begin{matrix} \text{open book} \\ \text{decompositions on } M \end{matrix} \right\} \Big/ \begin{matrix} \text{positive} \\ \text{stabilization} \end{matrix}$$

Theorem (Giroux 2000) The map ψ is onto

Theorem (Giroux 2002) The map ψ is one-to-one

Proof uses convex surface theory.