1 Introduction

Heegaard Floer homology $\hat{H}$, Khovanov homology $\tilde{K}$:

- Used in applications such as
  - Knot concordance
  - Dehn surgery
  - Contact geometry

- Different philosophies for theory's constructions
  - Ozsváth-Szabó discovered algebraic relationship between homologies

\[
\begin{align*}
\text{[Khovanov-type chain complex]} & \leftrightarrow \text{[Heegaard Floer-type chain complex assoc. to link]} \\
\text{[assoc. to link]} & \leftrightarrow \text{[to double-branched cover]}
\end{align*}
\]

For a link $L \subset S^3$ a spectral sequence whose

- $E^2$ term $= \tilde{K}(L)$
- $E^\infty$ term $= \hat{H}(\Sigma(S^3, L))$

Where $\tilde{K}$ denotes reduced Khovanov homology
$\hat{H}$ denotes mirror of $L$
$\Sigma(A,B)$ denotes double-branched cover of $A$ branched over $B$
$\hat{H}$ denotes Heegaard Floer homology

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NOTE: Unless explicitly stated all Khovanov & Heegaard Floer homologies have coefficients in \( \mathbb{Z}_2 \)

- Roberts building off of Plamenevskaya relationship more useful & general than originally thought

- Given a link \( L \) in the complement of a fixed unknot, \( BC \subset S^3 \)
  - \( \exists \) a spectral sequence from \( \text{Kh}^*(L) \) to a variant of knot Floer homology of \( B \subset \Sigma(S^3, L) \)
    - where \( B \) is preimage of \( B \) in \( \Sigma(S^3, L) \)

- Established relationship \( \text{Kh} \) \( \xrightarrow{\text{Plamenevskaya's transverse invariant}} \text{Osváth-Szabó contact invariant} \)

- Baldwin & Plamenevskaya used extension of relationship to establish tightness of a number of non Stein-fillable contact structures.

- Grigsby & Wehlau (different direction)

  Heegaard Floer homology for sutured manifolds (developed by Juhász)
  - simple variant of Khovanov homology categorifying the reduced, \( n \)-colored Jones polynomial
\[ \text{detects unknot whenever } n \geq 2 \]

Goal: Unify all these results

\text{NOTE: Framework uses Gabaï's sutured manifold theory and Juhász's sutured Floer homology}

Very nice! Can be shown to satisfy nice neutrality props. WRT certain TQFT operations

Let \( F \) be oriented surface w/ \( \partial F \neq \emptyset \), \( F \times I \) a product sutured manifold, \( T \subset F \times I \) a tangle (properly embedded 1-mfld) that is BOTH
- admissible: \( T \cap (\partial F \times I) = \emptyset \)
- balanced: \( |T \cap (F \times \{0\})| = |T \cap (F \times \{1\})| = n \in \mathbb{Z} \geq 0 \)

Then exists spectral sequence whose \( E^2 \) term = \( \text{Kh}^*(T) \)
\( E^\infty \) term = \( \text{SFH}(\Sigma (F \times I, T)) \)

\text{NOTE: The appropriate version of Khovanov homology for balanced tangles in product sutured manifolds is similar to [1] with abelianized...}
gradings where \( F = A \) (an annulus)
\( T = L \) is a 0-balanced tangle (link)

Main Theorems:

Thm 2.1: Let \( L \subset A \times I \) be a link in the product sutured mfld \( A \times I \), then there is a spectral sequence whose \( E^2 \) term is \( KH_*^*(L) \) whose \( E^\infty \) term is \( SFH(\Sigma(A \times I, L)) \).

**NOTE:** case when \( F = D \) in [6]

**NOTE:** Reinterpretation (slight extension) of Robert Main Result

Thm 1.1: [6, Prop 5.20] Let \( T \subset D \times I \) be an admissible balanced tangle. Then there is a spectral sequence whose \( E^2 \) term is \( KH_*^*(T) \) whose \( E^\infty \) term is \( SFH(\Sigma(D \times I, T)) \).

**NOTE:** Roberts restricts to \( L \) intersecting a spanning disk of \( B \) in odd # of pt
But Thm 1.1 makes NO restriction

**Connection:**

Let \( A \) be oriented annulus
\( I = [0,1] \) oriented closed unit inter
\( L \subset A \times I \) link

where \( A \times I \) identified as std sutured complement of standardly -imbeded unknot \( B \subset S^3 \) w/ identification

\[ \Delta \times T \subset \Sigma(A \times I, \partial_m) \]
Robert's constructs spectral sequence
\[ Kh^+( L) \rightarrow \text{Knot Floer homology} \]
(for links in)
(product mfld)
\( B \subset \Sigma(S^3, L) \) (\( B \) preimage of \( B \subset \Sigma \))
\[ \rightarrow \text{prop 2.24 shows that this } \]
\( \text{Knot Floer homology of } \widetilde{B} \text{ is the } \)
\( \text{sutured Floer homology of } \Sigma(A) \)

**NOTE:** nice relationship b/w spectral seqs
Thm 1.1 & Thm 2.1

\[ \rightarrow \text{A link } \text{LCA} \times I \text{ can be cut an } \]
\( \text{vertical disk to form admissible } \)
\( \text{balanced tangle } \text{TCD} \times I \)

**Thm 3.1:** Let \( \text{LCA} \times I \) be an isotopy class represen-
\( \text{of an annular link admitting a projectiv} \)
P(\( L \)) & let \( \lambda \text{ CA} \) be a properly imbedded
oriented arc representing a nontrivial element of $H_1(A,aA)$ s.t. $A$ intersect $P(L)$ transversely. Let $T \subset D \times I$ be the balanced tangle in $D \times I$ obtained by decomposing $A \times I$ (def 2.8) along the sur $\lambda \times I$ endowed w/ the product orientation.

Then the spectral sequence

$$Kh^\ast(T) \to SFH(\Sigma(D \times I, T))$$

is a direct summand of the spectral sequence

$$Kh^\ast(T) \to SFH(\Sigma(A \times I, L))$$

Moreover, the direct summand is trivial if $\exists L' \subset A \times I$ isotopic to $L$ satisfying

$$|1(A \times I) \cup L' | \triangleq |(A \times I) \cup L |$$

NOTE: 1\textsuperscript{st} example of "naturality" of relations btwn Kh & Heegaard Floer homology (under natural geometric operations) the spectral sequences behaves "as expected".

Interesting note: Given link $L \subset S^3$, any unknot $B \subset S^3 - N(L)$ endows the Khovanov chain complex associated to $L \subset B^3$ w/ $\mathbb{Z}$-filtration, via the identification
$S^3 - N(B) \leftrightarrow A \times I$

The extra grading inducing $\mathbb{Z}$-filtration has representation-theoretic interpretation.

Suppose $T \subset D \times I$ is an $n$-balanced tangle obtained by decomposing $L \subset A \times I$ along $\lambda \times I$. 

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