Brief intro to Legendrian and Transverse knots

Recall: A contact structure $\xi$ on $\mathbb{R}^3$ is a 2-plane field given locally as $\ker \alpha$ where $\alpha$ is a 1-form satisfying $\alpha \wedge \alpha \neq 0$.

Example: $\xi_{\text{std}} = \ker (dz - y \, dx) = \text{span} \{ \partial_z, \partial_x + y \partial_z \}$

Example: $\xi_{\text{sym}} = \ker (dz + r^2 \, d\theta) = \text{span} \{ \partial_r, r^2 \partial_z - 2 \partial \theta \}$

From Etnyre's Legendrian and Transverse Knots

Given a contact structure $\xi$ on $\mathbb{R}^3$, an embedding $i : S' \to \mathbb{R}^3$ we have 3 cases:

1. $i(S')$ is tangent to $\xi \to (T_x i(S') \subseteq \xi_x \ \forall x)$
2. $i(S')$ is transverse to $\xi \to (T_x i(S') \cap \xi_x = T_x \mathbb{R}^3 \ \forall x)$
3. $i(S')$ is sometimes tangent, sometimes transverse

We call 1 a Legendrian knot and 2 a transverse knot.
Equivalence of links:

Def: A Legendrian (Transverse) isotopy is an isotopy through a family of Leg. (Transverse) links

Fact: \textbf{3} Knots that are smoothly isotopic but not Leg. (transversely) isotopic

Fact: Any smooth link can be \(C^0\) approximated by a Legendrian link

Legendrian links (in \(\mathbb{R}^3\), \(\mathcal{E}_{\text{std}}\))

Front projection \(\Pi: \mathbb{R}^3 \rightarrow \text{xz-plane}\)

Lagrangian projection \(\tilde{\Pi}: \mathbb{R}^3 \rightarrow \text{xy-plane}\)

Properties of Front projections:

- no vertical tangencies \(dx = 0 \Rightarrow dz = 0\)
- recover y coord. by \(y = \frac{-dz}{dx}\)
- slope of overcrossing more negative

- Reid. moves given by

\(\longrightarrow \ X \ X \longrightarrow \bigg/ \ X \bigg/ \longrightarrow \ X \longrightarrow \bigg/ \ X \bigg/ \longrightarrow \bigg/ \)

\(R_1 \quad R_2 \quad R_3\)
Properties of Lagrangian Projections

- recover $z$-coord by $z = z_0 + \int_0^{2\pi} y(\theta)x'(\theta)\,d\theta$

- Must satisfy
  
  1. $\int_0^{2\pi} y(\theta)x'(\theta)\,d\theta = 0$
  
  2. $\int_0^{2\pi} y(\theta)x'(\theta)\,d\theta \neq 0$

- Partial Reidemeister moves:

  $K_1$ and $K_2$ are Legendrian isotopic only if their Lagrangian projections are related by $L_2$ and $L_3$.

  $L_2$ $\leftrightarrow$ $L_3$

Classical invariants of Legendrian links

- Thurston–Bennequin $\tau$ - in $(\mathbb{R}^3, \xi_{std})$ given by linking number of $L$ with small pushoff in $\xi$ direction.

  $tb(L) = \text{writhe}(\text{null}(L)) = -\frac{1}{2}(\# \text{ of cusps}) = \text{writhe}(\pi(L))$

Ex:

- $tb(L) = -1 \neq -2$  
- $tb(L) = -1 \neq -2$

Rotation $\tau$ - $r(L) = \frac{1}{2}(D-U) = \text{winding} \pi(L)$
Transverse links

Front projection:
- no downward vertical tangencies $\not\searrow$
- no crossings of the form $\nearrow$

Thus (2.9 in Etnyre) Any diagram satisfying the above 2 conditions gives a transverse knot in $(\mathbb{R}^3, \xi_{st})$. Two diagrams represent the same transverse isotopy class if and only if they are related by the moves below

$$
\begin{align*}
\begin{array}{c}
\text{T2} \\
\text{T3}
\end{array}
\end{align*}
$$

Ex: $\bigcirc = \bigcirc \neq \bigcirc$

Classical invt:

Self linking $sl(T) = \text{writhe } \nu(T)$

Transverse Links in $(\mathbb{R}^3, \xi_{sym})$  \(\xi_{sym} = \ker (dz + r \, d\theta)\)

Ex: $\bigcirc, \bigcirc, \bigcirc$
**Markov Moves for Transverse Links**

**Def:** A transverse link $L \in (\mathbb{R}^3, \xi_{sym})$ is a geometric braid if $\partial_{\xi} L > 0$.

**Thm (Bennequin, '83):** Any oriented transverse link is transverse isotopic to the closure of a braid.

**Thm (Orevkov and Shevchishin '02):** Two braids $B_1$, $B_2$ represent transversely isotopic links if and only if we can pass from $B_1$ to $B_2$ by conjugation, positive Markov moves and inverses.

**Notation:**
- $S_i = S'_1, \ldots, S'_n$
- Given $L : S \times I \rightarrow (\mathbb{R}^3, \xi_{sym})$, write $L_t = L(\cdot, t)$.
Def: A transversal isotopy $L: S \times I \rightarrow \mathbb{R}^3$ is monotone near the axis if $\exists \ t_i \ < \ \ldots \ < \ t_k \ \in \ I$ such that:

1) $\forall t: \exists! \ s_i \in S$ such that $L^{-1}(O_z) = \{(s_i, t_i), \ldots, (s_k, t_k)\}$

2) In every nbhd of $(s_i, t_i)$, $L$ is given by $x = t - 3s^2$, $y = st - s^3$, $z = z_i + s$ for $t$ coordinate on $I$ centered at $t_i$ and $s$ a coord. on $S$ centered at $s_i$.

$L$ is monotone everywhere if $L_t$ is a geometric braid for $t \notin \{t_i, \ldots, t_k\}$ and monotone near the axis.

Goal: Make every isotopy monotone everywhere.

Note: Fig. 1 represents a positive stabilization.
Steps:

1) Show \( L \) can be perturbed so as to be monotone near the axis.

2) Upgrade \( L \) to be monotone everywhere.

1) Replace every small nbhd of \( p = (s, t_i) \in \mathbb{I}^{-1}(O_\varepsilon) \) by fig 1. As long as \( U \) is sufficiently small, we can ensure that \( L_t \) is transverse.

Specifically, \( \frac{\partial^2}{\partial s^2} > \varepsilon \) near \( O_\varepsilon \) and can choose \( U \) small enough that \( r^2 \frac{\partial \Theta}{\partial s} < \varepsilon \) so that \( \frac{\partial^2}{\partial s^2} - r^2 \frac{\partial \Theta}{\partial s} |_{L_t} \to 0 \).

2) Need to make \( L_t \) a braid for \( t \neq t_i \).

Def: A **bad zone** of \( L \) is anywhere \( L \) is not a braid, i.e. any connected component in \( S \times I \) s.t. \( \Theta|_{L_t} \leq 0 \).

We call a bad zone \( V \) **simple** if

1) \( V_t := (S \times t) \cap V \) is connected for all \( t \in I \)

2) The total increment of \( \Theta \) along \( V \) is less than \( 2\pi \).
The shadow of $L$ on a bad zone $V$ is the set 
$\{ (s_0, t_0) \in V \mid \text{the shortest segment between } L(s, t) \text{ and } O_z \text{ intersects } L(\cdot, t_0) \text{ at } (s, t_0) \}$. 

Equivalently, points where $L(V)$ is an undercrossing in the $Oz$-projection.

**Lemma:** Can eliminate simple non-shadowed bad zones.

**Note:** z-coord fixed

**Lemma:** Can "wrinkle" a bad zone in a small nbhd $U$ of a smooth curve $\gamma \in V$ so that 
$\varepsilon > \frac{\partial \theta}{\partial s}/\frac{\partial z}{\partial s} > 0$ in that nbhd.

**Figure 3. Elimination of a bad zone (projection onto $Oxy$)**

**Figure 4. Wrinkling**

Morally: cut a bad zone along a smooth curve $c \times I$. 

(Ish)
Need to examine singularities of projection of \( L \) onto \( \Theta_2 \)-cylinder:

0. Generic singularities \( \leftrightarrow \) crossings
1. \( L_t \) meets \( z \)-axis (as in fig 1)
2. \( L_t \) has a unique ordinary tangency pt (T2)
3. \( L_t \) has a unique triple pt (T3)

A sing. is positive if \( \frac{2\theta}{2\pi} > 0 \) for every pt of \( L_t \) projecting onto it and non-positive o/w.

A sing. is bad if there is a negative arc shadowed by some other arc

Lemma: We can perturb all bad non-positive singularities of type (2) and (3)

![Figure 5. Elimination of bad non-positive singularities](image_url)

bad zones shadowed by bad zones
Algorithm:

1. Given $L$ monotone near the axis with bad zones $V_i, \ldots, V_n$, we eliminate bad zones successively via the following steps.

1') Eliminate bad non-positive singularities of Type (2) and (3).

Denote the shadows of $V_i$ on $V_j$ by $l_i, l_2, \ldots l_k$.

2) Wrinkle along components of bad zones $V_i$ shadowing $V_j$ as in fig. 6b.

3) Wrinkle $V_i$ wherever it's shadowed to get non-shadowed bad zones corresponding to $V_i$.

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Figure 6. Figure 7. Wrinkling at Step 2

Figure 8. Wrinkling at Step 3
4) Wrinkle new bad zones if needed to make sure they're simple.
5) Apply fig. 3 to get rid of non-shadowed bad zones.
6) Repeat for successive $V_i$.

Note: At each step, we wrinkle away from tangencies and triple pts, so we can make sure no new shadow appears.