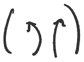
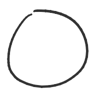
Meeting 1: (Hopf link) HOMFLY polynomial and Hilbert scheme of singular curve

Tuesday, January 10, 2023   2:26 PM

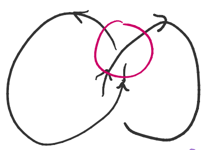
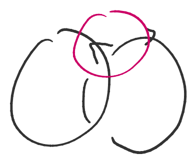
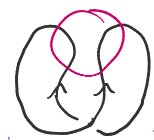
HOMFLY polynomial '80s

$L \longrightarrow P_L(a,q)$ → specialization $a=q^2$ → Jones polynomial; $a=1$ → Alexander poly

$a($  $) - a^{-1}($  $) = (q-q^{-1})($  $)$

Unknot  $= \dfrac{a-a^{-1}}{q-q^{-1}}$ $P_{L_1 \cup L_2} = P_{L_1} \cdot P_{L_2}$

Ex:

a  $- a^{-1}$  $= (q-q^{-1})$ 

H   2 unknots $\dfrac{(a-a^{-1})^2}{(q-q^{-1})^2}$ unknot $\dfrac{a-a^{-1}}{q-q^{-1}}$

$$aH - a^{-1}\frac{(a-a^{-1})^2}{(q-q^{-1})^2} = (q-q^{-1})\frac{a-a^{-1}}{q-q^{-1}}$$

$$aH = \left(\frac{a-a^{-1}}{q-q^{-1}}\right)\left(a^{-1}\frac{(a-a^{-1})}{q-q^{-1}} + (q-q^{-1})\right)$$

$$= \left(\frac{a-a^{-1}}{q-q^{-1}}\right)\left(\frac{1-a^{-2}+q^2-2+q^{-2}}{q-q^{-1}}\right)$$

$$= \left(\frac{a-a^{-1}}{q-q^{-1}}\right)\left(\boxed{\frac{q^2-1+q^{-2}}{q-q^{-1}}} - a^{-2}\right)$$

↳ most interesting for today

Consider

- q^2-1+q^{-2} (Laurent) polynomial in q three terms
- $\dfrac{q^2-1+q^{-2}}{q-q^{-1}} = q^{-1}\dfrac{1-q^2+q^4}{1-q^2} = q^{-1}\cdot\left(1+\dfrac{q^4}{1-q^2}\right)$

-1 / . 4 6 8 \

$$\frac{\quad}{q-q^{-1}} \quad \frac{\quad}{1-q^2} \quad \left(\frac{\quad}{1-q^2} \right)$$

$$= q^{-1}\underbrace{\left(1+q^4+q^6+q^8+\cdots\right)}_{\infty \text{ series } w/+ \text{ coeff}}$$

Khovanov - Rozansky

HOMFLY homology

$L \longrightarrow$ triply graded homology $HHH(L) = \bigoplus HHH^{i,j,k}(L)$

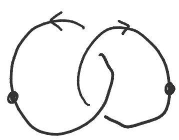
$$\sum a^i q^j \underbrace{(-1)^k}_{\text{Euler char}} \dim HHH^{ijk}(L) = P(a,q)$$

spectral seq. (Rasmussen)

Khovanov homology (considers harder)

$\hookrightarrow$ using Jones polynomial

Ex



positive Hopf link

machine

Hom deg 2 $\quad \mathbb{C}[x_1,x_2] \xrightarrow{0} \mathbb{C}[x_1,x_2] \xrightarrow{1} \mathbb{C}[x_1,x_2] \quad \longleftarrow$ a-degree 1

only homology

Hom deg 2 $\quad \mathbb{C}[x_1,x_2] \xrightarrow{0} \mathbb{C}[x_1,x_2] \xrightarrow{x_2-x_1} \mathbb{C}[x_1,x_2] \quad \longleftarrow$ a-degree 0 $\quad$ corresponds to q^2-1+q^{-2}

Hom deg 0

$\frac{1}{(1-q^2)^2}$

$$H^* = \underbrace{\frac{\mathbb{C}[x_1,x_2]}{(x_1-x_2)}}_{\text{compute generating series, graded ring}} \rightarrow \text{after grading } \frac{1}{1-q^2}$$

Exercise: Compute graded dim of H^* assuming $\deg_j(x_1) = \deg_j(x_2) = 2$

$\longrightarrow$ see actual

Remark: 1) Module over $\mathbb{C}[x_1,x_2]$ $\longleftarrow$ natural link invariant

marked point on link components $\longleftrightarrow$ action of $\mathbb{C}[x]$

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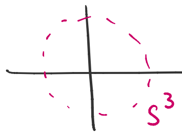
2) S_2 action!

Triply graded homology models

couple cases known homotopic

- $\hookrightarrow$ Hilb^k(singular curves) & Affine Springer fibers — Algebraic links
- $\hookrightarrow$ braid varieties — pos. braids
- $\hookrightarrow$ $\mathrm{Hilb}^k(\mathbb{C}^2)$ & $\underbrace{\text{coherent sheaves}}_{\text{information stored}}$ — can add full twist (NICE)
  - $\hookrightarrow$ just tensored w/ line bundle

Ex: $\{xy=0\} \subset \mathbb{C}^2$



$C = \{xy=0\} \cap S^3 =$ 2 circles

$\mathbb{C}[x,y] =$ ring of alg. fcn on $\mathbb{C}^2$

$$\frac{\mathbb{C}[x,y]}{(xy)} = \underbrace{\mathcal{O}_C}_{\text{ring of alg. fcns on } C} \qquad \frac{\mathbb{C}[[x,y]]}{(xy)} = \underbrace{\mathcal{O}_{C,0}}_{\text{completed local ring of fcn on } C \text{ at } 0}$$

(xy): ideal defining curve

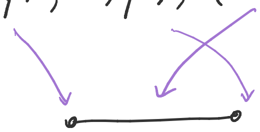
$\mathrm{Hilb}^k(C,0) =$ Hilbert scheme of k points on C at 0

$$\| \quad \left\{ I \subset \mathcal{O}_{C,0} : I \text{ ideal } \dim \frac{\mathcal{O}_{C,0}}{I} = k \right\}$$

$k=0$ $\quad \mathcal{O}_{C,0}$

$k=1$ $\quad \mathfrak{m} = (x,y)$

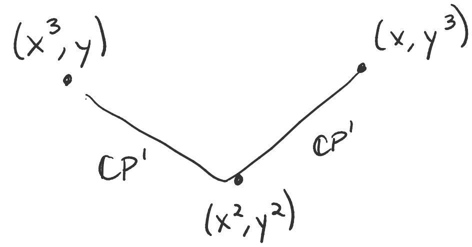
$k=2$ $\quad (x^2,y),\ (x,y^2),\ (\alpha x+\beta y)\quad \alpha,\beta \neq 0$



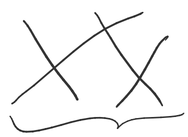
$\mathbb{CP}^1$

$k=3$ $\quad (x^3,y) \qquad\qquad (x,y^3)$

$k=3$



Conclusion:

k	Hilb^k
0	•
1	•
2	$\mathbb{CP}^1$
3	 $2\mathbb{CP}^1$
$\vdots$	
k	 chain of $(k-1)$ $\mathbb{CP}^1$

$H^0 = \mathbb{C}$

$H^2 = \mathbb{C}^k$

Claim/Exercise:

$$\bigoplus_{k=0}^{\infty} \underbrace{H^*(\mathrm{Hilb}^k)}_{\substack{\text{naturally bigraded by } k \\ \text{\& homological degree}}} = \left(\begin{matrix} a=0 \\ \text{part} \end{matrix}\right) \text{ of } HHH\left(\begin{matrix}\text{Hopf} \\ \text{link}\end{matrix}\right)$$

$$\mathbb{C}[x_1, x_2] \xrightarrow{0} \mathbb{C}[x_1, x_2] \xrightarrow{x_1 - x_2} \mathbb{C}[x_1, x_2]$$

Remark:
- S_2 action (swap x & y)
- Subtle: action of x_1, x_2 (Rennemo, Kivinen, ...)
- "Full" Hilbert scheme $(k+1)$-components

$$Bl_{\mathbb{C}^{j-1} \times \mathbb{C}^{k-i-1}} \mathbb{C}^i \times \mathbb{C}^{k-i}$$

$$B|_{\mathbb{C}^{j-1}\times\mathbb{C}^{k-i-1}} \quad \mathbb{C}^{i}\times\mathbb{C}^{k-i}$$