Meeting 2: Define Khovanov-Rozansky Homology

Tuesday, January 17, 2023 2:59 PM

Sections: 3.1, 3.2 (in lecture notes)

1) Bimodules

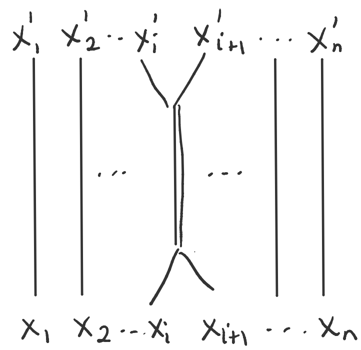
$R = \mathbb{C}[x_1, \dots, x_n]$

number of strands in a braid

coefficients, could be $\mathbb{Z}$

$$B_i = \frac{\mathbb{C}[x_1, \dots, x_n, x_1', \dots, x_n']}{\begin{pmatrix} x_i + x_{i+1} = x_i' + x_{i+1}' \\ x_i x_{i+1} = x_i' x_{i+1}' \\ x_j = x_j' \quad j \neq i, i+1 \end{pmatrix}} (1) \qquad 1 \leq i \leq n-1$$

shift grading by 1



This is an R-R bimodule:
left R acting on x_i
right R acting on x_i'

Remark: 1) $f(x_i, x_{i+1}) = f(x_i', x_{i+1}')$ for any symmetric fcn f.
2) Graded, $\deg(x_i) = 2$

$*\deg(f(x,x')1) = \deg(1) + 2d$

def: A Bott-Samelson bimodule is a product
$B_{i_1} \otimes_R B_{i_2} \otimes_R \cdots \otimes_R B_{i_k}$

Equivalently, concatenate pictures

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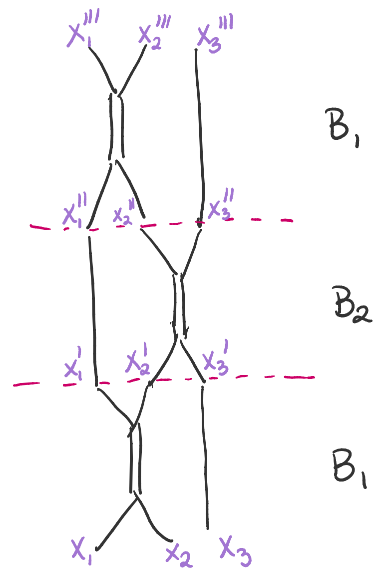
e.g. $B_1 \otimes_R B_2 \otimes_R B_1$



"web"

Fact: There are bimodule maps

a) $b_i : B_i(-1) \longrightarrow R \quad b_i(1) = 1 \qquad R = \dfrac{\mathbb{C}[x_1,\dots,x_n,x_1',\dots,x_n']}{x_j = x_j' \quad \forall j}$

grading

b) $b_i^* : R \longrightarrow B_i(1) \quad b_i(1) = \underbrace{x_i - x_{i+1}'}_{deg=2}$

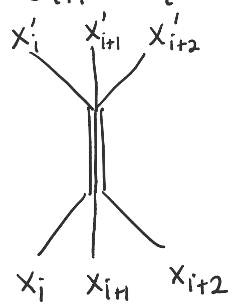
$$b_i^*(x_i - x_i') = (x_i - x_i')(x_i - x_{i+1}') = (x_i - x_i)(x_i - x_{i+1}) = 0$$

symmetric in x_i', x_{i+1}' — in B_i

Similarly, $b_i^*(x_{i+1} - x_{i+1}') = 0$

Fact: 1) lemma 3.4: $B_i \otimes B_i \simeq B_i(1) \oplus B_i(-1)$

2) Ex 3.5: $B_i \otimes B_{i+1} \otimes B_i \simeq B_i \oplus B_{i,i+1}$



$f(x_i, x_{i+1}, x_{i+2})$
$= f(x_i', x_{i+1}', x_{i+2}')$
for any symmetric f

def: Rouquier complexes

$$T_i = [B_i(-1) \xrightarrow{b_i} R]$$
$$T_i^{-1} = [R \xrightarrow{b_i^*} B_i(1)]$$

complexes of R-R bimodules

Thm: (Rouquier) T_i, T_i^{-1} satisfy braid relations: (up to homotopy)

1) $T_i \otimes_R T_i^{-1} = T_i^{-1} \otimes_R T_i \simeq R$

2) $T_i \otimes T_{i+1} \otimes T_i \simeq T_{i+1} \otimes T_i \otimes T_{i+1}$

3) $T_i \otimes T_j \simeq T_j \otimes T_i \qquad |i-j| \geq 2$

Remark: General formula

$$M = [\cdots \longrightarrow M_i \xrightarrow{d_M} M_{i-1} \longrightarrow \cdots]$$
$$N = [\cdots \longrightarrow N_j \xrightarrow{d_N} N_{j-1} \longrightarrow \cdots]$$

$$M \otimes N = \bigoplus_{i,j} \underbrace{M_i \otimes N_j}_{i+j}$$

$$\begin{tikzcd}
 & M_{i-1} \otimes N_j \\
M_i \otimes N_j \arrow[ur, "d_M \otimes 1"] \arrow[dr, "(-1)^i \otimes d_N"'] & \\
 & M_i \otimes N_{j-1}
\end{tikzcd}$$

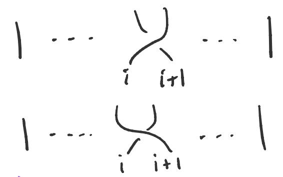
proof idea: (lemma 3.11 in lecture notes)

$$T_i \otimes T_i^{-1} = [B_i(-1) \xrightarrow{b_i} R] \otimes [R \xrightarrow{b_i^*} B_i(1)]$$

$$= \left[\begin{tikzcd}
 & R \arrow[dr, "b_i^*"] & \\
B_i(-1) \arrow[ur, "b_i"] \arrow[dr, "-b_i^*"'] & \oplus & B_i(1) \\
 & B_i \otimes B_i \arrow[ur, "b_i"'] &
\end{tikzcd}\right] \simeq (R)$$

contractible by lemma 3.4

Q: How does this relate to braids?

$T_i \rightsquigarrow$ 

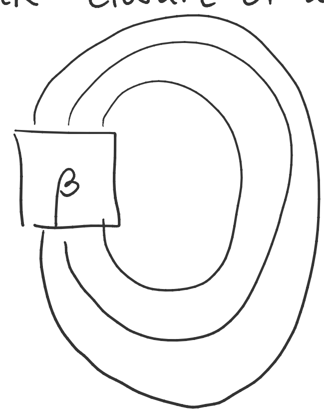
$T_i^{-1} \rightsquigarrow$

crossings

Cor: For <u>any</u> braid β, we can define a complex T_β of R-R bimodules by multiplying T_i, T_i^{-1}
T_β is well defined up to homotopy equivalence

↳ "<u>Rouquier complex associated</u> to β"
pick a braid $\longrightarrow$ $\boxed{\text{machine}}$ $\longrightarrow$ chain of complexes

Knot/link = closure of a braid



Q: what does closure mean?

- M = R-R bimodule

$$\left.\begin{array}{l} HH^0(M) = \underset{\text{bimodule maps}}{\mathrm{Hom}}(R,M) \end{array}\right\} \text{most important}$$

$$HH^i(M) = \mathrm{Ext}^i(R,M)$$

$\beta \longrightarrow$ Rouquier complex T_β $\longrightarrow$ $\underbrace{\text{apply } HH^0 \text{ or } HH^i \text{ term-wise}}$ $\longrightarrow$ take homology

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Result = HOMFLY homology $HHH(\beta)$

 $\rightsquigarrow B_i \longrightarrow R \rightsquigarrow HH^0(B_i) \longrightarrow HH^0(R)$

<u>Three gradings</u>:

1) "Hochschild" grading $= i$ for HH^i
   ($\sim a$ in HOMFLY)
2) Homological (from Rouquier complex)
3) Quantum ($\sim q$ in HOMFLY)
   $\deg(x_i) = 2$