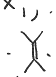
Meeting 4: Homology on Two Strands using Rouquier Complex

Monday, January 30, 2023 3:10 PM

Recap:

$R = \mathbb{C}[x_1, \dots, x_n]$

$B_i = | \cdots$  $\cdots |$ (i, $i+1$)

R-R bimodule

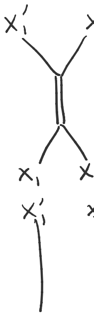
$T_i = [B_i^{(-1)} \xrightarrow{b_i} R]$
$T_i^{-1} = [R \xrightarrow{b_i^*} B_i^{(1)}]$
} complexes of R-R bimodules

β = braid $\longrightarrow$ Rouquier complex T_β
$= \otimes$ of T_i, T_i^{-1} for crossings

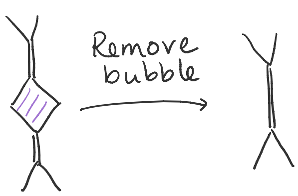
Ex: $n=2$

$$B_1 = B = \frac{\mathbb{C}[x_1, x_2, x_1', x_2']}{\begin{pmatrix} x_1 + x_2 = x_1' + x_2' \\ x_1 x_2 = x_1' x_2' \end{pmatrix}}$$

$$R = \mathbb{C}[x_1, x_2] = \frac{\mathbb{C}[x_1, x_2, x_1', x_2']}{\begin{pmatrix} x_1 = x_1' \\ x_2 = x_2' \end{pmatrix}}$$



Fact: $\underset{B \otimes_R B}{\overset{\|}{B^2}} \simeq B(1) \oplus B(-1)$

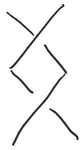


Ex: Any braid on two strands
$= (\times)^K \quad K \in \mathbb{Z}$

$K=0$  R

$K=1$  $\rightsquigarrow T = [B(-1) \longrightarrow R]$

$K=1$  $\rightsquigarrow T = [B(-1) \longrightarrow R]$

$K=2$  $\rightsquigarrow T^2 = [B(-1) \to R] \otimes [B(-1) \to R]$

$$= \begin{tikzcd}[row sep=small] & B(-1) \arrow[dr] & \\ B \otimes B(-2) \arrow[ur] \arrow[dr] & & R \\ & B(-1) \arrow[ur] & \end{tikzcd}$$

NOTE: $M \otimes_R R = M$
$\quad = R \otimes_R M$

$$= \begin{tikzcd}[row sep=small] & B(-1) \arrow[dr] & \\ {[B(-1) \otimes B(-3)]} \arrow[ur] \arrow[dr] & & R \\ & B(-1) \arrow[ur] & \end{tikzcd}$$

$\wr\wr$ homotopy equiv!

$$T^2 \simeq [B(-3) \xrightarrow{x_1 - x_1'} B(-1) \xrightarrow{b} R]$$

$*$ $b((x_1 - x_1')\cdot f = (x_1 - x_1')\cdot b(f) = 0$ in R $\Rightarrow d^2 = 0$

$\to$ so this is a well-defined complex

Claim: $T^K \simeq [\underbrace{B \to B \to \cdots \xrightarrow{x_1 - x_2'} B(-7) \xrightarrow{x_1 - x_1'} B(-5) \xrightarrow{x_1 - x_2'} B(-3) \xrightarrow{x_1 - x_1'} B(-1)}_{K} \xrightarrow{b} R]$
$K \geq 0$

$$(x_1 - x_2')(x_1 - x_1') \underset{\uparrow}{=} (x_1 - x_2)(x_1 - x_1) = 0$$
symmetric in x_1', x_2'

Recall: $f(x_1', x_2') = f(x_1, x_2)$ f symmetric in 2 variables

proof: Induction. $\square$

$K < 0$: $R \xrightarrow{b_1^*} \underbrace{B(1) \xrightarrow{x_1 - x_1'} B(3) \xrightarrow{x_1 - x_2'} B(5) \xrightarrow{x_1 - x_1'} \cdots}_{|K|}$

NOTE: We have duals
$B^\vee = B$, $R^\vee = R$, $T^\vee = T^{-1}$, $(T^\vee)^K = T^{-K}$

To close braid, 2 steps:

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- Apply $HH^i = R^i\mathrm{Hom}(R,-)$ termwise
- Take homology $HH^0 = \mathrm{Hom}_{bimod}(R,-)$ ("a degree 0")

Aside: Q: what is a dual?

Category $\mathcal{C}$, $\otimes$

X = object

$X^\vee$ = another object, dual to X

$X \otimes X^\vee \longrightarrow \mathbb{1}$

$\mathbb{1} \longrightarrow X^\vee \otimes X$

Ex: $X = V$ vector space

$X^\vee = V^*$

$$\underset{\mathrm{End}(V)}{\underset{\parallel}{V \otimes V^*}} \xrightarrow{Tr} \mathbb{C}$$

$$\mathbb{C} \xrightarrow[\text{identity matrix}]{} V \otimes V^*$$

- $HH^0(R) = \mathrm{Hom}_{bimod}(R,R) = R$ generated by id
- $HH^0(B) = \mathrm{Hom}(R,B) = R$ generated by b^*

Case 1: $K \geq 0$

$$\underbrace{\cdots \xrightarrow{0} R \xrightarrow{x_1 - x_2} R \xrightarrow{0} R}_{K} \xrightarrow{x_1 - x_2} R$$

$x_1 - x_1' = 0$ in R

Ex: $B \longrightarrow R$

$\uparrow b^*$

R

Apply HH^0

$b \circ b^* = x_1 - x_2' \underset{\text{in } R}{=} x_1 - x_2$

$$\mathrm{Hom}(R,B) \longrightarrow \mathrm{Hom}(R,R)$$

$$R \ni \varphi \qquad\qquad R \qquad b \circ \varphi$$

$$b^* \longrightarrow b \circ b^*$$

Recall: $b^*(1) = x_1 - x_2'$

$b(b^*(1)) = (x_1 - x_2') = x_1 - x_2$ in R

Homology of complex:

$$\begin{array}{lccccccccc}
 & & H^{K-1} & & & H^3 & H^2 & H^1 & H^0 \\
K \text{ odd:} & 0 & \dfrac{\mathbb{C}[x_1,x_2]}{(x_1-x_2)} & 0 & \cdots & 0 & \dfrac{\mathbb{C}[x_1,x_2]}{(x_1-x_2)} & 0 & \dfrac{\mathbb{C}[x_1,x_2]}{(x_1-x_2)} = \mathbb{C}[x]
\end{array}$$

$\hookrightarrow$ $T(2,K)$ knot (1-component)

$$\begin{array}{lccccccc}
K \text{ even:} & \underset{R}{\mathbb{C}[x_1,x_2]} & 0 & \cdots & 0 & \dfrac{\mathbb{C}[x_1,x_2]}{(x_1-x_2)} & 0 & \dfrac{\mathbb{C}[x_1,x_2]}{(x_1-x_2)}
\end{array}$$

$\hookrightarrow$ $T(2,K)$ link (2-component)

⚠ Very important observation:
For $K \geq 0$, $HHH^0(T(2,K))$ (in fact, all HHH^i) are in <u>even</u> homological degrees

Case 2: $K<0$

Apply HH^0

$$\underbrace{\underset{\operatorname{Hom}(R,R)}{R_0} \xrightarrow{1} \underset{\operatorname{Hom}(R,B)}{R_{-1}} \xrightarrow{0} R_{-2} \xrightarrow{x_1-x_2} R_{-3} \xrightarrow{0} R_{-4} \xrightarrow{x_1-x_2} \cdots R \xrightarrow{x_1-x_2} \underbrace{R}_{K-1} \xrightarrow{0} \underbrace{R}_{K}}_{|K|}$$

$$\begin{array}{ccc}
R & \xrightarrow{b^*} & R \\
\uparrow 1 & & \\
R & &
\end{array}$$

b^* generates $\operatorname{Hom}(R,B)$

$T = B_1 \longrightarrow R_0$

$T^{-1} = R_0 \longrightarrow B_{-1}$

Homology of complex

$$\begin{array}{ccccccc}
H^0 & H^{-1} & H^{-2} & H^{-3} & H^{-4} & H^{-5} & \cdots \\
0 & 0 & 0 & \dfrac{\mathbb{C}[x_1,x_2]}{(x_1-x_2)} & 0 & \dfrac{\mathbb{C}[x_1,x_2]}{(x_1-x_2)} &
\end{array}$$

$$\begin{array}{lc}
 & H^{-K} \\
K \text{ odd:} & \dfrac{\mathbb{C}[x_1,x_2]}{(x_1-x_2)}
\end{array}$$

$$\begin{array}{lccc}
 & H^{-(K-2)} & H^{-(K-1)} & H^{-K} \\
K \text{ even:} & 0 & \dfrac{\mathbb{C}[x_1,x_2]}{(x_1-x_2)} & \mathbb{C}[x_1,x_2]
\end{array}$$