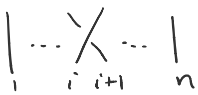
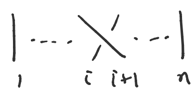
Meeting 5

Monday, February 6, 2023 3:13 PM

Recap:

$B_i =$ 

$T_i =$ 

$T_i^{-1} =$ 

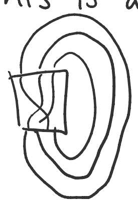
β braid $\rightsquigarrow$ product of $T_i^{\pm}$ = complex of

T_β R-R bimodules

$HHH(\beta) = R\,\mathrm{Hom}(R, T_\beta)$

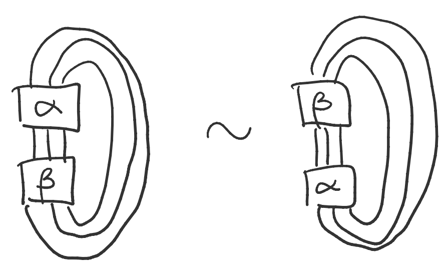
<u>Properties:</u>

1) This is a link invariant!

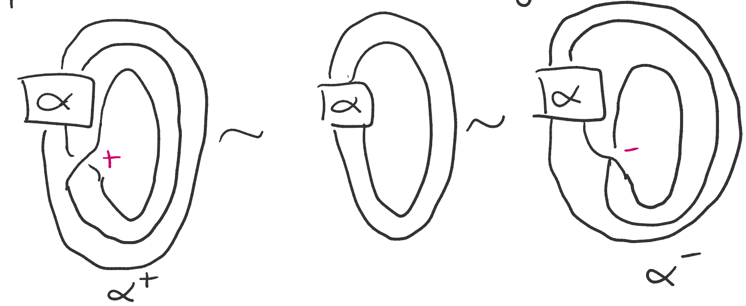
β  $\bar{\beta}$ = closure of β

<u>Idea</u>: Markov Thm: Any two braids representing same link are related by a sequence of moves:

M1) $\alpha\beta \sim \beta\alpha$



M2) positive stabilization & negative stabilization



<u>Fact</u>:

1) Exercise: M, N R-R bimodules

$$R\mathrm{Hom}(R, M\otimes N) \simeq R\mathrm{Hom}(R, N \otimes M)$$

↳ as a vector space

$\Longrightarrow$ (M1) is true

2) Positive stabilization preserves all degrees

$$R\mathrm{Hom}^i(\alpha) = R\mathrm{Hom}^i(\alpha^+)$$

Negative stabilization <u>shifts</u> all degrees, in particular

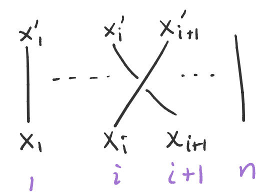
$$R\mathrm{Hom}^i(\alpha) \approx R\mathrm{Hom}^{i+1}(\alpha^-)$$

Cor: $\mathrm{Hom}(R, T_\beta)$ = "$a=0$" part of HHH is invariant under positive stabilization but NOT under negative stabilization.

3) R-action

lemma: For $T_i^{\pm}$, the action of x_i is homotopic to the action of x'_{i+1}

$x_{i+1} \sim x'_i$



Recall: C° = chain complex

$f, g : C^\circ \longrightarrow C^\circ$ (chain maps $df = fd$ & $dg = gd$)

are chain homotopic if there is h

s.t. $dh \pm hd = f - g$

($\Longrightarrow f, g$ act same way in $H_*(C^\circ)$)

proof (lemma):

$T_i = B_i \underset{b_i^*}{\overset{b_i}{\rightleftarrows}} R$

$$\begin{tikzcd}
\left[B_i \arrow[r, "b_i"] \arrow[d, shift right, "x_i'"'] \right. & \left. R \right] \arrow[d, shift right, "x_i"'] \arrow[dl, "b_i^*"'] \\
\left[B_i \arrow[r, "b_i"] \arrow[u, shift right, "x_{i+1}'"'] \right. & \left. R \right] \arrow[u, shift right, "x_{i+1}"']
\end{tikzcd}$$

$b_i(1 \otimes 1) = 1$

$b_i^*(1) = x_i - x_{i+1}'$

Can use $b_i^* = h$ as the homotopy btwn

x_i & x_{i+1}'

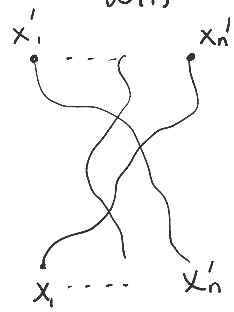
$b_i b_i^* = b_i^* b_i = x_i - x_{i+1}'$ $\square$

Cor: For any β, action of x_i is homotopic to $x'_{w(i)}$

where w = permutation for β.

If we close the braid

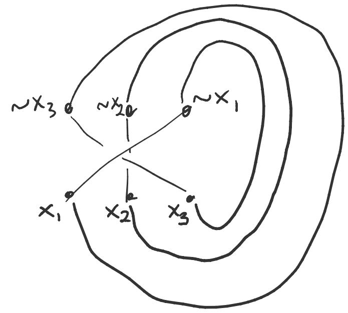
$x'_{w(1)} = x_{w(1)} \sim x_i$



Thm: If $L = L_1 \cup \dots \cup L_r$ is a link w/ r components

then $HHH(L)$ is naturally a module over

r variables (one for each components)



- $x_1 \sim x_3$
- x_2

$\}$ variables

Last time:

$T(2,k)$ $\quad k>0$ odd

HHH:

$$\frac{\mathbb{C}[x_1,x_2]}{x_1-x_2} \quad 0 \quad \frac{\mathbb{C}[x_1,x_2]}{x_1-x_2} \quad 0 \quad \cdots \quad \frac{\mathbb{C}[x_1,x_2]}{x_1-x_2}$$

$x_1 = x_2$ on homology (knot w/ 1 component)

k even:

extra $\mathbb{C}[x_1,x_2]$

Fact: (Rasmussen)

If K is a knot then $HHH(K) = \mathbb{C}[x_1] \otimes \overline{HHH}(K)$ ↳ Reduced HHH

& $\overline{HHH}$ is finite dim'l

Ex: $\overline{HHH}^{a=0}(T(2,2k+1)) = H_*(\mathbb{C}P^k)$

Thm: (Elias, Hogancamp, Mellit)

If L = positive torus link $T(m,n)$, then $HHH(T(m,n))$ is supported in <u>even degrees</u>

Thm: (Hogancamp, E. Gorsky)

$T(m,n)$ torus link n components, pairwise linked  bad drawing

$HHH^{a=0}(T(n,n)) = J/yJ$

where $J = \bigcap_{i \neq j} \underbrace{(x_i - x_j, y_i - y_i)}_{\text{ideals}} \subset \mathbb{C}[x_1, \ldots, x_n, y_1, \ldots, y_n]$