Meeting 3: Cluster Structures on Braid Varieties (James)

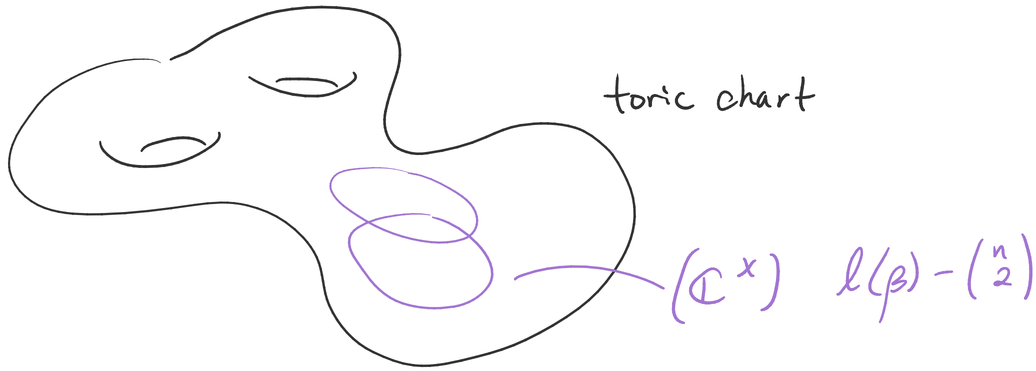
Monday, April 17, 2023 3:10 PM

NOTE: Most conventions will differ from Eugene's

Recall: Given $\beta \in Br_n^+$, $X(\beta) := \{ z_1, \dots, z_{\ell(\beta)} : w_0 B_{\sigma(1)}(z_1) \cdots B_{\sigma(\ell(\beta))}(z_{\ell(\beta)}) \text{ is upper triangular} \}$

$$B_i(z_j) = \begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & z_j & 1 & \\ & & 1 & 0 & \\ & & & & \ddots \\ & & & & & 1 \end{pmatrix} \qquad \beta = \sigma(1) \cdots \sigma(\ell(\beta))$$

Cluster Structures on Braid Varieties



Facts: 1) For $\beta = \cdots \sigma_i \sigma_i \cdots$

$$X(\beta) = \mathbb{C}^\times \times X(\cdots \sigma_i \cdots) \sqcup \mathbb{C} \times X(\cdots)$$

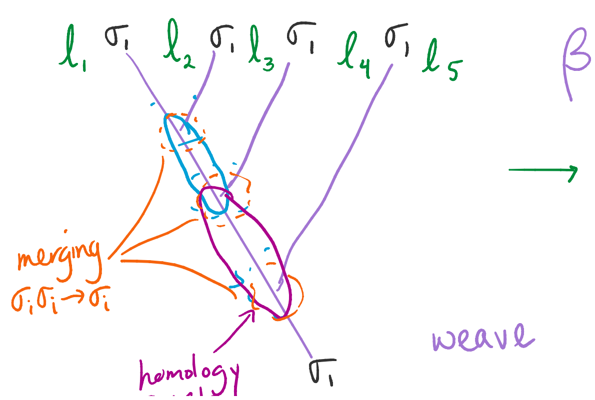
2) $X(\Delta_n) = \mathbb{C}^{\binom{n}{2}}$

pos. braid lift of w_0

① & ② $\Longrightarrow$ can obtain a toric chart $(\mathbb{C}^\times)^{\ell(\beta)-\binom{n}{2}}$ by repeated iterations of

$$\cdots \sigma_i \sigma_i \cdots \longrightarrow \cdots \sigma_i \cdots$$

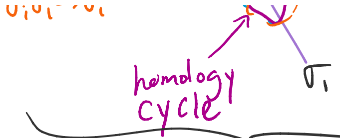
Ex: $\beta = \sigma_1 \sigma_1 \sigma_1 \sigma_1$



$\longrightarrow$ $\ell_1 \pitchfork \ell_3$ $\longrightarrow$ $v_1 \wedge v_3$

$\ell_1 \pitchfork \ell_4$ $\quad$ $\underbrace{v_1 \wedge v_4}$

give a coordinate basis for $(\mathbb{C}^\times)^2$ induced by $(\mathbb{C}^\times)^2 \times X(\Delta) \hookrightarrow X(\sigma^4)$



$\underbrace{\hspace{3em}}$ Legendrian in $\mathbb{R}^5$ projects to a Lagrangian in $\mathbb{R}^4$

$\longrightarrow V_1 \wedge V_3 = z_1$

$V_1 \wedge V_4 = [B_1(z_1)\, B_1(z_2)] = 1 + z_1 z_2$

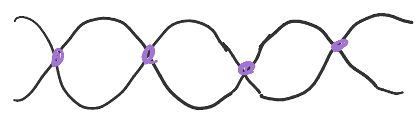
$(\mathbb{C}^*)^2$ induced by

$(\mathbb{C}^*)^2 \times X(\Delta) \hookrightarrow X(\sigma_1^4)$

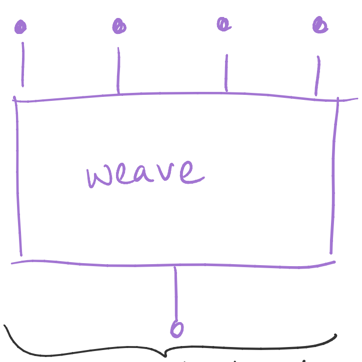
where

$V_i \wedge V_j = [B_{\sigma(i)}(z_1) \cdots B_{\sigma(j-2)}(z_{j-2})]_{(1,1)}$

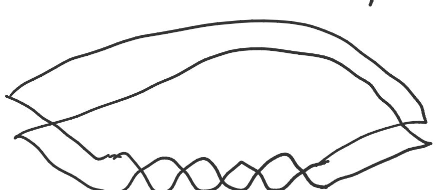
Algebraic Weaves



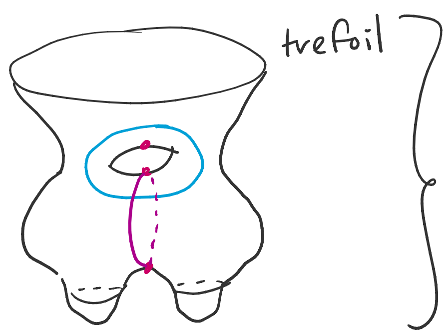
σ_1^4



5 σ_1's at bdry

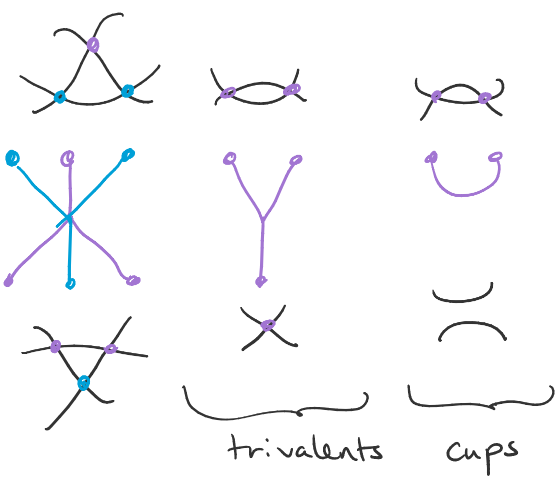


crossing convention



Lagrangian in $\mathbb{R}^4$ from weave

<u>Weave moves</u>



NOTE: Y $\neq$ U

Thm: [CGGS] A weave from β to β' induces a morphism

$\mathbb{C}^a \times (\mathbb{C}^\times)^b \times X(\beta') \hookrightarrow X(\beta)$

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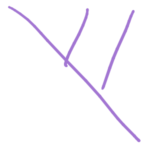
$$\mathbb{C}^a \times (\mathbb{C}^\times)^b \times X(\beta') \hookrightarrow X(\beta)$$

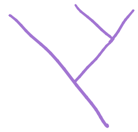
where $a = \#$ of cups

$b = \#$ of trivalents

Q: How to go between toric charts?



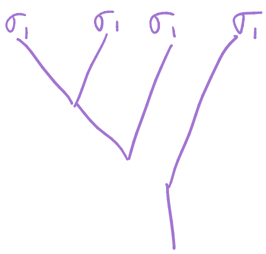
$\neq$



$(\sigma_i \sigma_i)\sigma_i$ $\qquad$ $\sigma_i(\sigma_i\sigma_i)$

} mutation moves

Q: How is this related to stratification?



$$X(\beta) = (\mathbb{C}^\times)^2 \sqcup \mathbb{C} \sqcup \mathbb{C}$$

$z^2 + 2$

$a=0$ term of HOMFLY poly

NOTE: Objects : $X(\beta)$

morphisms : maps btwn weaves

$\exists$ Functor $\mathfrak{X}$: Weaves $\longrightarrow \mathcal{C}$ s.t. Thm [CGGS] above