• Presentation document for Feb13

We define the family of polynomials $p(v.w)$ as follows:

$v.w$: binary sequence
with $|v| = |w| = \#$ of 1 in $v = \ell$

① $p(\phi.0^n) = p(0^n.\phi) = \left(\frac{1+a}{1-q}\right)^n$

② $p(v1, w1) = (t^{\ell}+a)\, p(v.w)$

③ $p(v0, w1) = p(v, 1w)$

④ $p(v1.w0) = p(1v, w)$

⑤ $p(v0.w0) = t^{-\ell} p(1v, 1w) + q t^{-\ell} p(0v, 0w)$

Our main takeaway is that

the triply graded Khovanov-Rozansky homology of $T(m.n)$ $(:= HHH(T(m.n)))$ is free over $\mathbb{Z}$ of graded rank $p(0^m.0^n)$, and so

$HHH(T(m.n))$ is supported in even degrees.

Ex $p(0^2.0^2)$

$$p(00,00) = p(10,10) + q\,p(00.00)$$

So, $p(00.00) = \frac{1}{1-q}\, p(10,10)$.

- $p(10,10) = t^{-1} p(11,11) + q t^{-1} p(01.01)$

Here, $p(11.11) = (t+a)(1+a)$ and

$p(01.01) = 1+a$

thus $p(00,00) = \frac{1+a}{1-q}\{t+a+1\}$.

Important Combinatorial Fact about $p(0^m.0^n)$

- $p(0^m.0^n)|_{a=0}$ = $(m.n)$ rational q,t Catalan number

<u>Thm</u> (Hogancamp-Mellit)

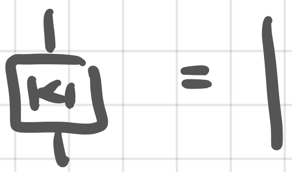
HHH(T(m.n)) is supported in even degrees with its rank $P(0^m, 0^n)$.

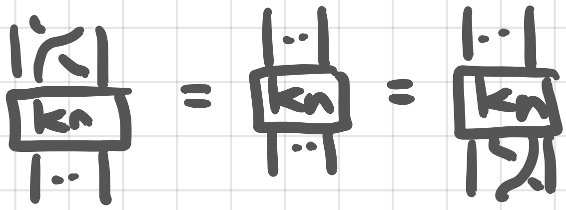
We'll sketch the idea of this theorem.

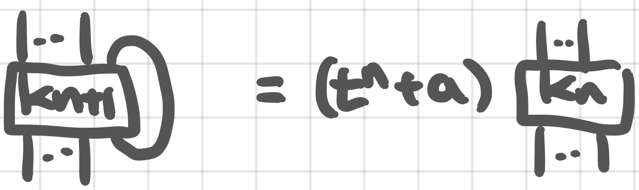
- <u>Idea of proof</u>

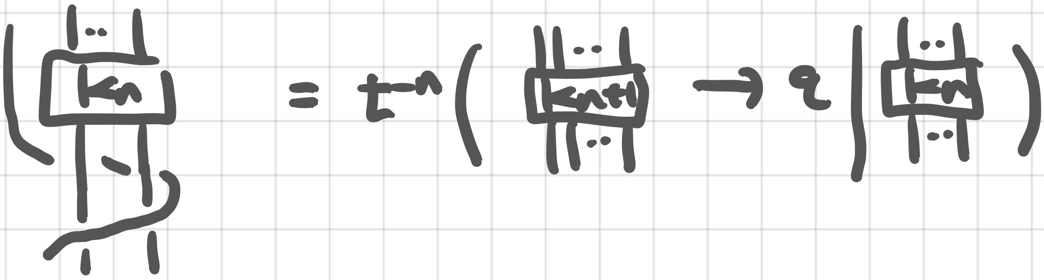
① <u>Thm</u> (Hogancamp)

There are complex of Soergel bimodules K_n satisfying the following relations:

(1) 

(2) 

(3) 

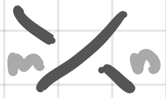
(4) 

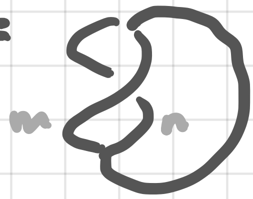
② Diagrammatics

Recall that we've pictured the complexes in the following diagrammatical way.

- Diagram for a strand 

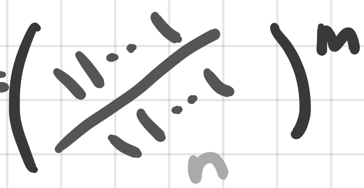
- Diagram for torus links

$X_{m.n} :=$  braid

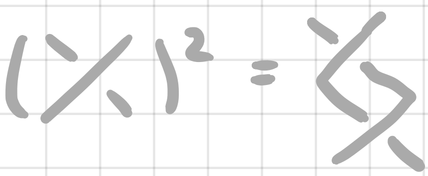
$T(m.n) :=$  Torus link

※ Caution

We use the different notation for $T(m.n)$ in our lecture note.

$T(m.n) :=$ 

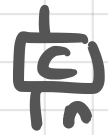
So, this braid (& link) consists of n strands.

For example, $T(2.2) =$ 

However, the notation above has m+n strands.

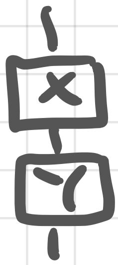
Actually, these two definition coincides if we apply Markov moves, but showing this in general is nontrivial.

- Diagram for a complex

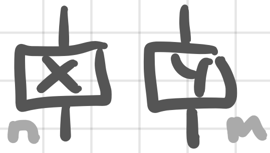
A complex C corresponds to a diagram 

Tensor product operation on two complex X and Y

corresponds to a concenating of two diagram

 .

Similarly, the external tensor product operation $X \sqcup Y$ are indicated by



<u>Note</u> Actual statement was that

$$K_n \otimes F(\beta) \simeq K_n \simeq F(\beta) \otimes K_n$$

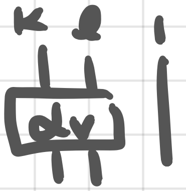
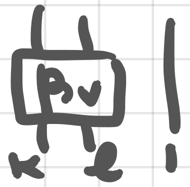
$$HH((X \sqcup \mathbb{1}_1) \otimes K_{n+1}) \simeq (t^n + a)\, HH(X)$$

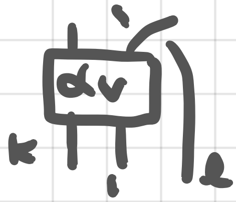
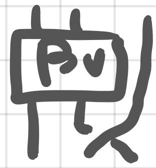
$$(\mathbb{1}_1 \sqcup K_n) \otimes L_{n+1} \simeq t^{-n}(K_{n+1} \to q(\mathbb{1}_1 \sqcup K_n))$$

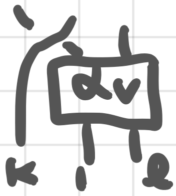
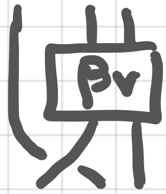
We also had a diagrammatic descriptions of the positive braid list of π_v and π_v^{-1}.

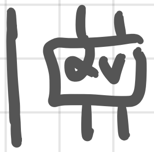
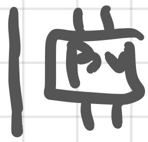
$\alpha_v :=$ positive braid list of π_v

$\beta_v :=$ negative braid list of $\pi_{v^{-1}}$

$\alpha_{v1} =$  vs $\beta_{v1} =$ 

$\alpha_{v0} =$  vs $\beta_{v0} =$ 

$\alpha_{1v} =$  vs $\beta_{1v} =$ 

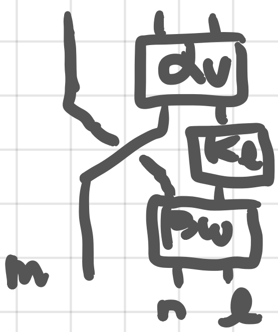
$\alpha_{0v} =$  vs $\beta_{0v} =$ 

And we defined a complex

$C(v.w) \in K^b(\mathbb{BS}_{m+n+\ell})$ by

$$(\mathbb{1}_n \sqcup F(\alpha_v)) \otimes (F(X_{m.n}) \sqcup K_\ell) \otimes (1_n \sqcup F(\beta_w))$$

So, this new complex $C(v.w)$ will be depicted by



③ Sketch of proof

Then we prove our main theorem by showing that

- $HH(C(V,W))$ satisfies the categorical analogues of the reccursion which defines $P(V,W)$ i.e

$$HH(C(V1,W1)) \simeq (t^{\ell}+a)\, HH(C(V,W))$$

$$HH(C(V0,W1)) \simeq HH(C(V,1W))$$

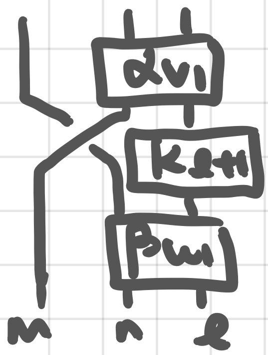
$$HH(C(V1,W0)) \simeq HH(C(1V,W))$$

$$HH(C(V0,W0)) \simeq t^{-\ell}\left(HH(C(1V,1W)) \to q\, HH(C(0V,0W)\right)$$

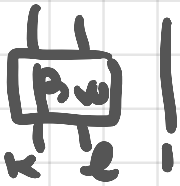
For example, we check that

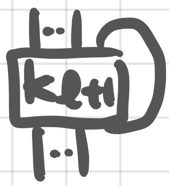
$$HH(C(V1,W1)) \simeq (t^{\ell}+a)\, HH(C(V,W)).$$

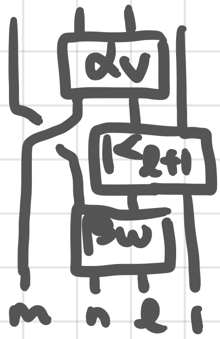
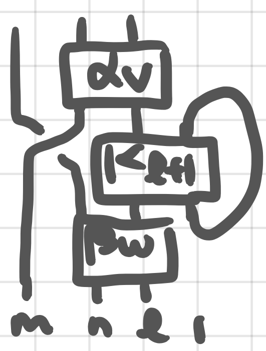
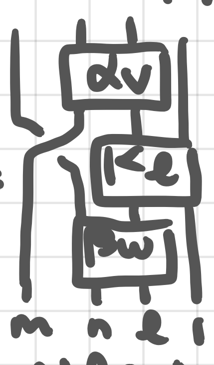
Since $C(V1,W1)$ is pictured as



and $\alpha_{V1} =$ 

$\beta_{W1} =$ 

 $= (t^{\ell}+a)$ 

we have $C(V1,W1) =$  $=$  $=$  $\cdot\, (t^{\ell}+a)$

- Then Prop 4.12 in "On the Computation of Torus link Homology" (ElTos)

tells us that applying HH before and after tensoring with K_1 are related by taking $- \otimes_{\mathbb{Z}} \mathbb{Z}[x]$.

In other words,

$$
\begin{aligned}
HH(C(0^m, 0^n)) &\simeq HH(C(10^{m-1}, 10^{n-1})) \otimes_{\mathbb{Z}} \mathbb{Z}[x] \\
&\simeq \frac{1}{1-q} P(10^{m-1}, 10^{n-1}) \mathbb{Z} \\
&= p(0^m, 0^n) \mathbb{Z}.
\end{aligned}
$$

- How do we know that $HHH(T(m,n))$ is supported in even degrees?

Our polynomial $p(0^m, 0^n)$ are polynomial in three variables q, t and a.

We use the change of variables

$q = Q^2$

$t = T^2 Q^{-2}$

$a = A Q^{-2}$

Here, Q was the homological degree, so this finalizes our claim!

Note In the proof of [65], we use the Induction on (v,w) and divided the cases

when ① $(v',w') = (0^m, 0^n)$

② $(v',w') = (v1, w1)$

③ $(v',w') = (v1, w0)$

④ $(v',w') = (v0, w1)$

⑤ $(v',w') = (v0, w0)$

But in the last case, since

$$HH(C(v',w')) \simeq [t^{-\ell} HH(C(1v, 1w)) \to q^{-\ell} HH(0v, 0w))$$

to show that $HH(C(v',w'))$ is actually supported in even homological degrees, we need the following additional lemma.

Lem Suppose that we have an exact triple of complexes

$$0 \to A_\bullet \to B_\bullet \to C_\bullet \to 0$$

and the homology of $A_\bullet$ & $C_\bullet$ is supported in even homological degrees. Then the homology of $B_\bullet$ is supported in even homological degrees and

$$H_k(B_\bullet) = H_k(A_\bullet) \oplus H_k(C_\bullet) \quad \forall k.$$

Whenever we have a twisted differential map

$\delta : A[-1] \to B$ that shifts degree by 1,

we have a $\mathrm{Cone}(\delta) := A \oplus B$ and so

we have a short exact sequence

$$0 \to B \to \mathrm{Cone}(\delta) \to A \to 0 .$$

Since we have

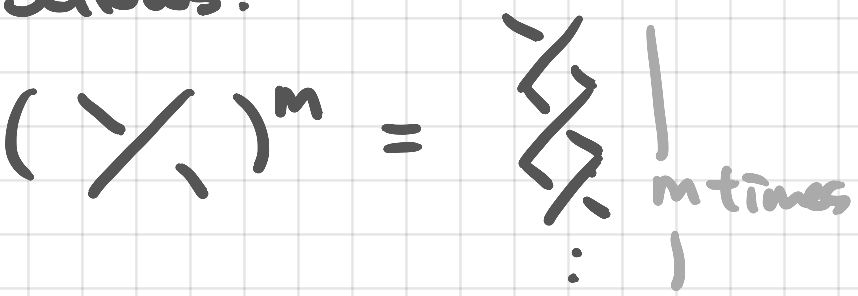
$$HH(C(v', w')) \simeq \left[t^{-\ell} HH(C(1v, 1w)) \to q^{-\ell} HH(0v, 0w) \right)$$

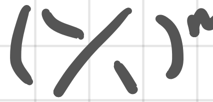
Showing that $HH(C(1v, 1w))$, $HH(0v, 0w)$ is supported in purely even homological degrees this implies that $HH(C(v', w'))$ is also evenly supported.

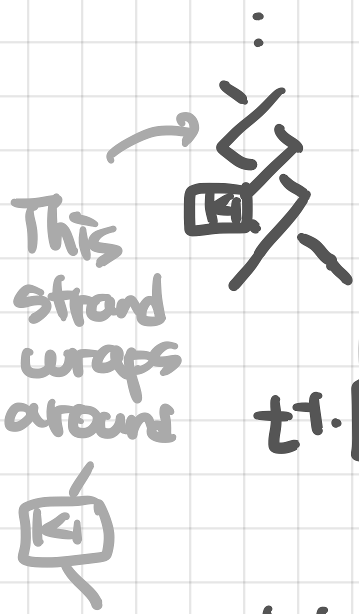
Ex 3.2.6

Let's compute the homology of two strand torus links HHH(T(2, m)) !

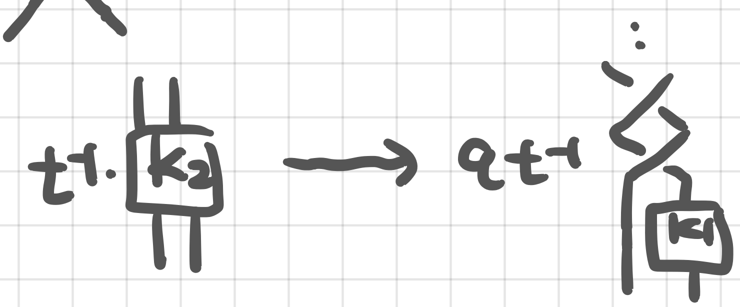
If we use the notations from our main reference, we can describe the braid of T(2, m) as follows.



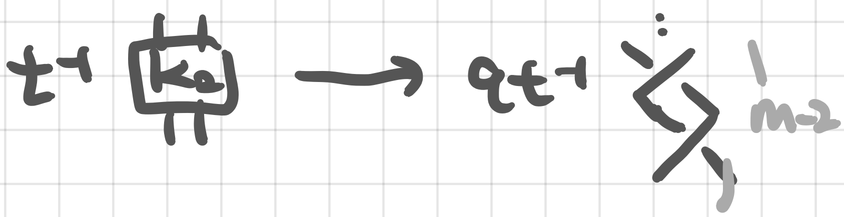
Using the construction of kn,  is homotopic equivalent to



Then (4) implies that this is equal to



which is same as



Thus $HHH(T(2,m)) = \dfrac{t^{-1}(t+a)(1+a)}{1-q} + HHH(T(2,m-2))$

Rmk

1) Plugging in $a=0$, we get the formula for $HHH^0(T(m,n))$ stated in Ex 3.23.

2) This idea can't be used for 3 strands torus link, so when the given torus link has 3 or more strands,
we rely on the recursion of $p(v,w)$.
(So, in this case,  notation stated in [65] is a lot better!)