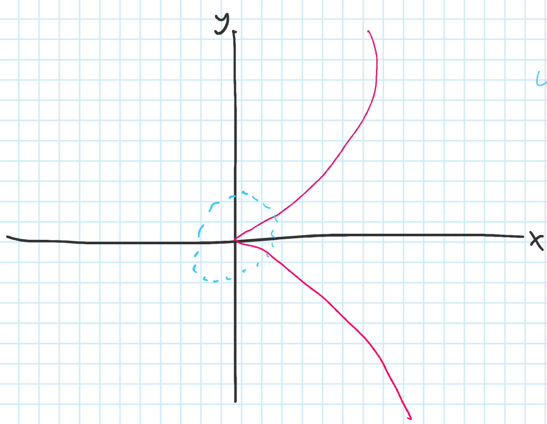
Next, we would like to give yet another interpretation of $\mathrm{Hilb}^n(C,0)$ using geometric representation theory. Let us choose a projection of C to some line, and let n be the degree of this projection. We will regard the line as a local model for the "base curve" and C as a "spectral curve".

Remark 6.7. *The choice of the projection naturally splits the unit sphere in $\mathbb{C}^2$ as a union of two solid tori. Indeed, the equation of the sphere is $|x|^2+|y|^2=\varepsilon^2$ and the solid tori are $|x|^2 \le \frac{\varepsilon^2}{2}$ and $|y|^2 \le \frac{\varepsilon^2}{2}$. For ε small the intersection of C with a sphere defines a closed n-strand braid which is contained in one of the tori. This is known as a braid monodromy construction, see e. g. [2] and references therein.*

$$|x|^2 \le \frac{\varepsilon^2}{2}, \qquad |y|^2 = \varepsilon^2 - |x|^2$$

$$D^2 \times S^1$$



Here, $|y|$ is small, somehow we get a braid

We will use the following results.

Lemma 6.8. *Let C be a germ of an arbitrary plane curve (possibly non-reduced) given by the equation $\{f(x,y)=0\}$.*

(a) One can replace $f(x,y)$ by a polynomial of some degree n in x with coefficients given by power series in y.

(b) A (topological) basis in $\mathcal{O}_{C,0}$ is given by monomials of the form $x^a y^b$, $a \le n-1$. In other words, $\mathcal{O}_{C,0}$ is a free $\mathbb{C}[[y]]$–module of rank n with basis $1,\ldots,x^{n-1}$.

(c) The multiplication by x and y in this basis is given by the matrices:

$$Y \mapsto \begin{pmatrix} y & 0 & 0 & \cdots & 0 \\ 0 & y & 0 & \cdots & 0 \\ 0 & 0 & y & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & y \end{pmatrix}, \qquad X \mapsto \begin{pmatrix} 0 & 0 & 0 & \cdots & -f_0(y) \\ 1 & 0 & 0 & \cdots & -f_1(y) \\ 0 & 1 & 0 & \cdots & -f_2(y) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & -f_{n-1}(y) \end{pmatrix}$$

In particular, the characteristic polynomial of the second matrix equals $\det(X - x\cdot I) = f(x,y)$.

Pf: (a) In $\mathbb{C}[[x]]$, we can always write

$$f = x^n u \quad \text{for } n = \operatorname{ord}(f), \ u \text{ a unit.}$$

The Weierstrass preparation thm is a generalization of this, that if A is a complete local ring, then this, that if A is a complete local ring, then any $f \in A[[x]]$ can be written (if not all coef. are in m)

$$f = uF, \text{ where}$$

$$F = b_0 + b_1 x + \dots + b_{n-1}x^{n-1} + x^n$$

and $b_i \in m \subseteq A$ the unique maximal ideal.

So if $A = \mathbb{C}[[y]]$, $m = (y)$, then (if $y \nmid f$)

$f = uF$, and $(f) = (F)$, so we can replace f with

$$F = x^n + f_{n-1}(y)x^{n-1} + \dots + f_0(y).$$

(b): $\mathcal{O}_{C,0} = \mathbb{C}[x,y]/F$, so we have the relation

$$x^n = -f_{n-1}(y)x^{n-1} - \dots - f_0(y). \text{ So}$$

$\mathcal{O}_{C,0}$ is a free $\mathbb{C}[[y]]$ module w/ basis

$1, x, \dots, x^{n-1}$, (n is the degree of the projection to the y-axis)

and $x^a y^b$, $a \le n-1$ is a basis for $\mathcal{O}_{C,0}$ over $\mathbb{C}$.

(c): $y(x^a y^b) = x^a y^{b+1}$, and

$$x(x^a y^b) = \begin{cases} x^{a+1}y^b & \text{if } a < n-1, \text{ or} \\ (-f_{n-1}(y)x^{n-1} - \dots - f_0(y))y^b & \text{if } a = n-1 \end{cases}$$

Note that x and y are not symmetric, this all depends on our choice of projection.

Example 6.10. *For the cusp $C = \{x^2 = y^3\}$ we have $\mathcal{O}_{C,0} = \mathbb{C}[[x]]\langle 1, y, y^2\rangle$ so that*

$$Y = \begin{pmatrix} 0 & 0 & x^2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

On the other hand, we can choose a different projection and write $\mathcal{O}_{C,0} = \mathbb{C}[[y]]\langle 1, x\rangle$ so that

$$X = \begin{pmatrix} 0 & y^3 \\ 1 & 0 \end{pmatrix}.$$

In both cases the characteristic polynomial equals (up to sign) $x^2 - y^3$.

We will use Lemma 6.8 to give a description of $\mathrm{Hilb}^N(C, 0)$ when $N \gg 0$ and C is irreducible, see also Section 6.4 below. First, let us recall that for the group SL_n the **affine Grassmannian** is the ind-variety

$$Gr_{SL_n} := SL_n(\mathbb{C}((x)))/SL_n(\mathbb{C}[[x]]).$$

The affine Grassmannian Gr_{SL_n} has the following interpretation. A **lattice** $V \subseteq \mathbb{C}((x))^n = \mathbb{C}^n((x))$ is a free $\mathbb{C}[[x]]$-submodule of rank n such that $V \otimes_{\mathbb{C}[[x]]} \mathbb{C}((x)) = \mathbb{C}^n((x))$. In other words, a lattice V is the $\mathbb{C}[[x]]$-span of a $\mathbb{C}((x))$-basis $(v_1, \ldots, v_n)$ of $\mathbb{C}^n((x))$. Let us say that a lattice V is of SL_n-type if we can find such a basis so that the determinant of the matrix with columns $v_1, \ldots, v_n$ is 1. It is known then that the affine Grassmannian parametrizes such lattices,

$$Gr_{SL_n} = \{V \subseteq \mathbb{C}^n((x)) : V \text{ is a lattice of } SL_n\text{-type}\}.$$

Remark 6.11. *Of course, one can do a similar construction with GL_n instead of SL_n, and obtain that the affine Grassmannian $Gr_{GL_n} = GL_n(\mathbb{C}((x)))/GL_n(\mathbb{C}[[x]])$ parametrizes all lattices in $\mathbb{C}^n((x))$.*

Now define the affine Springer Fiber

$$Sp_\gamma = \{ g \in Gr_{SL_n} \mid g^{-1}\gamma g \in SL_n(\mathbb{C}[[x]]) \}$$

$$= \{ \Lambda \in Gr_{SL_n} \mid \gamma\Lambda \subseteq \Lambda \}$$

This matrix γ comes from $f(x,y)$ using Lemma 6.8 and has $\mathrm{chp}(y) = f(x,y)$

(X)

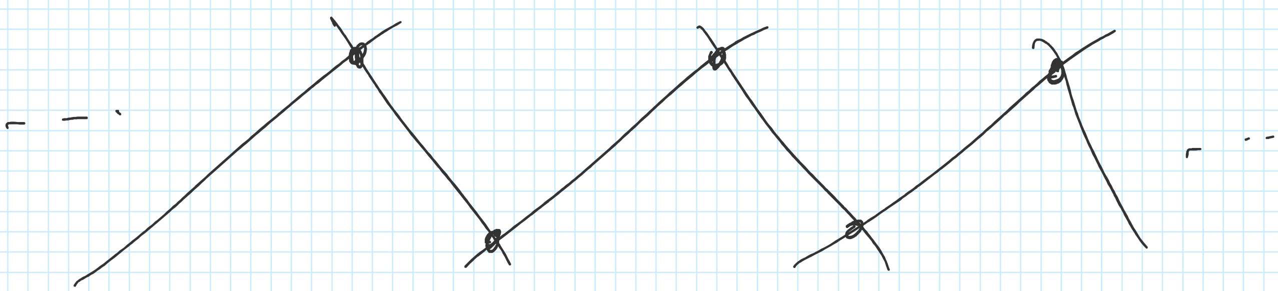
Rmk: (a) If $(C,0)$ is irreducible, then $Sp_\gamma \cong \mathrm{Hilb}^N(C,0)$ for $N \gg 0$. In particular, it's a projective variety.

(b) If $(C,0)$ has r irreducible components, then Sp_γ is an ind-variety with infinitely many irreducible components. There is a lattice action by $\mathbb{Z}^{r-1}$ and an action of $(\mathbb{C}^*)^{r-1}$.

Ex: $C = \{x^2 = y^2\}$, so

$$\gamma = \begin{pmatrix} 0 & x^2 \\ 1 & 0 \end{pmatrix} \quad \left(\sim \begin{pmatrix} x & 0 \\ 0 & -x \end{pmatrix} \right).$$

Then Sp_γ is a chain of $\mathbb{P}^1$'s:



$\mathbb{Z}$ acts by translations, $\mathbb{C}^*$ acts by stretching all the $\mathbb{P}^1$'s.

Homology is (roughly)

$\mathbb{C}[y] \quad \mathcal{O} \quad \mathbb{C}$

Theorem 6.14 ([81, 86]). *One has*

$$\bigoplus_{k=0}^{\infty} H^*(\mathrm{Hilb}^k(C,0)) = \mathrm{gr}_P H^*(\mathrm{Sp}_Y) \otimes \mathbb{C}[x],$$

where gr_P refers to the associated graded with respect to a certain "perverse" filtration on the cohomology of Sp_Y.

Furthermore, there is an action of $\mathfrak{sl}_2$ on $H^*(\mathrm{Sp}_Y)$ satisfying "curious hard Lefshetz" property with respect to the perverse filtration.

The ORS conjecture is that

$$HHH^0(L) = \bigoplus_{k=0}^{\infty} H^*(\mathrm{Hilb}^k(C,0))$$

In the case where $(C,0)$ is irreducible, Sp_γ is isomorphic to the compactified Jacobian of C, which is either

(a) The space of rank 1 torsion free sheaves "of degree 0" on C

(b) The moduli space of $\mathcal{O}_{C,0}$ submodules of $\mathbb{C}[[t]]$ up to a shift by powers of t. Here we parametrize $f(x,y) = f(x(t), y(t))$ so that

$$\mathcal{O}_{C,0} = \mathbb{C}[[x(t), y(t)]]$$

Ex: If $C = \{x^m = y^n\}$ $(\gcd(m,n)=1)$, then

$$\mathcal{O}_{C,0} = \mathbb{C}[[t^n, t^m]].$$

Eugene will work through this example in detail next week.

Consider $C = \{x^{kn} = y^n\}$, which corresponds to the (n, kn) torus link. The matrix for this

Consider $C = (x \quad y)$, which corresponds

to the (n, kn) torus link. The matrix for this is conjugate in $SL_n(\mathbb{C}((t)))$ to

$$\gamma = \begin{pmatrix} z_1 x^k & & \\ & \ddots & \\ & & z_n x^k \end{pmatrix}$$

notes said n?

where z_i are the distinct n^{th} roots of unity (or just distinct and nonzero). Oscar Kivinen found that (roughly)

$$H^T_{*,BM}(S_{p_\gamma}) = \bigcap_{i<j} (x_i - x_j, y_i - y_j)^k$$

as a module over $\mathbb{C}[x_1, \ldots, x_n, y_1, \ldots, y_n]$.

$f = \prod (z_i x^{d_i} - y)$

I found that for

$$\gamma = \begin{pmatrix} z_1 x^{d_1} & & \\ & \ddots & \\ & & z_n x^{d_n} \end{pmatrix},$$

that (roughly)

$$H^T \quad (S \quad) \quad \bigcap \quad (\quad)^{d_{ij}}$$

$$H^T_{*,BM}(Sp_\gamma) = \bigcap_{i<j} (x_i - x_j, y_i - y_j)^{d_{ij}}$$

where $d_{ij} = \min(d_i, d_j)$. The precise relationship to K-R homology is:

Theorem 6.24 ([73]). *(a) One has* This is for (n,kn) torus link

$$H_{*,BM}(\mathrm{Sp}_Y) = \mathrm{HHH}^0(T(n,kn)) \bigotimes_{\mathbb{C}[x_1,\ldots,x_n]/(\prod_i x_i - 1)} \mathbb{C}[x_1^{\pm},\ldots,x_n^{\pm}]/\left(\prod_i x_i - 1\right)$$

where the action of x_i on the left hand side is given by the lattice $\mathbb{Z}^{n-1}$, and on the right hand side by Theorem 5.10.

(b) Similarly,

Localize action $\qquad \Delta$

$$H^T_{*,BM}(\mathrm{Sp}_Y) = \underline{\mathrm{HY}^0(T(n,kn))} \bigotimes_{\mathbb{C}[x_1,\ldots,x_n]/\prod_i x_i - 1} \underline{\mathbb{C}[x_1^{\pm},\ldots,x_n^{\pm}]}/\underline{\left(\prod_i x_i - 1\right)}$$

where $T = (\mathbb{C}^*)^{n-1}$ and the equivariant parameters $y_1,\ldots,y_n$ with $\sum_i y_i = 0$ match the ones appearing in the y-ification on the right. One can avoid the restrictions to the codimension 1 subtori by considering the GL_n-affine Springer fibers instead.

y-ification