Meeting 9: Review

Monday, March 13, 2023 3:13 PM

C = irreducible plane curve through $(0,0)$, e.g. 

Fact: We can always parametrize C (Newton)

$$x(t) = t^n$$

$$y(t) = t^m + \underbrace{\cdots}_{\text{higher order terms}}$$

power series

$\rightarrow$ If $t = x^{1/n}$

$$y(x) = x^{m/n} + \ldots$$

Fact: The affine Springer fiber corresponding to

$C = \mathrm{Hilb}^N(C)$ for $N >> 0$

$$= \left\{ \begin{array}{l} V \subset \mathbb{C}((t)) \\ x(t)\, V \subset V \\ y(t)\, V \subset V \end{array} \right\}$$

$\longrightarrow$ proof (ask Josh Turner)

Ex: $x(t) = t^2$
$y(t) = t^3$

$$\left\{ \begin{array}{l} V \subset \mathbb{C}((t)) \\ t^2 V \subset V \\ t^3 V \subset V \end{array} \right\} \quad (\star)$$

If $\varphi(t) \in \mathbb{C}((t))$ w/ ord $\varphi(t)$ = smallest degree of monomial in φ w/ nonzero coeff.

Can assume minimal order of $\varphi(t) \in V$ is 0

$$V = (1 + \lambda t, t^2, t^3, \ldots) \qquad \lambda \in \mathbb{C}$$

-or-

$$V = (1, t, t^2, \ldots)$$

$\boxed{Sp_\gamma = \mathbb{CP}^1}$ where γ curve describing C

Q: Meaning of minimal order?

$$V = (t^k + \lambda t^{k+1}, t^{k+2}, \ldots)$$

$$V = (t^k, t^{k+1} \ldots)$$

$V = \breve{\gamma}(t^k + \lambda t^{k+1}, t^{k+2}, \ldots)$

$V = (t^k, t^{k+1}, \ldots)$

NOTE: If v satisfies $(\star)$ the $t^k V$ also satisfies $(\star)$ for $k \in \mathbb{Z}$

up to this "shift"

Can assume min. order $= 0$

Thm: (Piontkowski)

Suppose $x = t^n$, $y = t^m + \ldots$, $\gcd(m,n) = 1$

link $= (m,n)$ torus knot

So Spr has a cell decomposition

Cells $(\cong \mathbb{C}^d) \longleftrightarrow (m,n)$ invariant subsets

All even-dim'l, $\dim_{\mathbb{C}}(\text{cell}_\Delta) = \sum_i \#[a_i, a_{i+1}) \setminus \Delta$

where $a_i =$ n-generators of Δ

NOTE: $\boxed{\text{cell}_\Delta = \{V : \Delta_V = \Delta\}}$ $*$

def: $\Delta \subset \mathbb{Z}$ is (m,n)-invariant if $\Delta + m \subset \Delta$, $\Delta + n \subset \Delta$

a_i is an n-generator of Δ if $a_i \in \Delta$ & $a_i - n \notin \Delta$

Remark: In any remainder mod n we have exactly 1-generator

Ex: $(m,n) = (2,3)$

dim $= 1 \to \underline{0}, \underline{2}, \underline{3}, \underline{4}, 5, \ldots$ } 2,3 invariant subset — 3-generators

dim $= 0 \to \underline{0}, \underline{1}, \underline{2}, 3, 4, 5, \ldots$ — 2-generators

If $\min(\Delta) = 0$, there are exactly 2 invariant subsets

lemma: If $x(t)V \subset V$, $y(t)V \subset V$, define $\Delta_V = \{\text{Ord}\, \varphi(t) : \varphi(t) \in V\} \subset \mathbb{Z}$,

then Δ_V is (m,n) invariant

Lemma: If $x(t)V\subset V$, $y(t)V\subset V$, define $\Delta_V=\{\mathrm{Ord}\,\varphi(t):\varphi(t)\in V\}\subset\mathbb{Z}$,
then Δ_V is (m,n) invariant

Cor: (Thm) We can compute $H_*(Sp_\gamma)$ & similarly, $\underbrace{H_*(\mathrm{Hilb}^k(C))\ \forall k}_{(\mathrm{ORS})}$

Remark: (G, Mazin, Oblomkov)

$\underbrace{x=t^n,\ y=t^m+\lambda t^{m+1}+\dots,\ \lambda\neq 0}_{}$, where (m,n) not necessarily coprime link = certain specific cable of $(\frac{m}{d},\frac{n}{d})$ torus knot, $d=\gcd(m,n)$

e.g. (t^4, t^6+t^7)

In this case, we still have cell decomposition

cells = (m,n)-invariant subset & additional condition

dim = same formula

Q: How to prove that $H_*(Sp_\gamma)$ is related to HHH?

- Hogancamp - Mellit:

  HHH can be computed using some recursion (Soyeon's talk)

Claim: $H_*(Sp_\gamma)$ satisfies same recursion

Idea: $u\in\{0,1\}^{m+n}$ ← binary sequence $\quad \Delta\subset\mathbb{Z}_{\geq 0}$

$$P_u=\sum_{\substack{\Delta\ (m,n)\text{-invariant}\\ \Delta\cap[0,m+n-1]=u}} q^{|\mathbb{Z}_{\geq 0}-\Delta|}\,t^{\dim\square *}$$

Then P_u satisfies $\quad P_{0v}=q(P_{v0}+P_v)$

$\qquad P_{1v}=t^{\cdots}P_{v1}$

proof: • If $0\notin\Delta$, we can consider $\Delta-1$. This changes $(\mathbb{Z}_{\geq 0}-\Delta)$ by 1 does not change dim.

• If $0\in\Delta$, can erase Δ & consider $(\Delta\setminus\{0\})-1$

by 1 does not change dim.

- If $0 \in \Delta$, can erase Δ & consider $(\Delta \setminus \{0\}) - 1$

  $m \in \Delta,\ n \in \Delta,\ m+n \in \Delta$

Can say how dim changes

Ex: $\cancel{0}$ _ 2 3 4 5 $\quad \dim = 1$ (# of spaces)

1 2 3 4 5 $\quad \dim = 0$