Talk 3/6/23

Lemma from last time, basically for $C=\{f=0\}$

$$\mathcal{O}_{C,0} \cong \mathbb{C}[[y]]\langle 1, x, x^2, \dots, x^{n-1}\rangle \cong \mathbb{C}^n[[y]]$$
$$(=(\mathbb{C}[[y]])^n$$

where y acts as Y and

$$x \text{ acts as } X = \begin{pmatrix} 0 & & & -f_0(y) \\ 1 & \ddots & & \vdots \\ & \ddots & 0 & \\ & & 1 & -f_{n-1}(y) \end{pmatrix} \quad (\mathrm{ch}_X(x) = f(x,y))$$

$(\mathrm{Hilb}^n(C,0))$

Ideals in $\mathcal{O}_{C,0}$ = lattices in $\mathbb{C}^n((y))$
that are
- invariant under y-action,
- invariant under X-action
- contained in $(\mathbb{C}[[y]])^n$

If $\gamma = X$, then
$Sp_\gamma :=$ lattices in $\mathbb{C}^n((y))$
- Invariant under y, (Gr)
- Invariant under γ

(Shows: up to mult. by y^k)

so we just drop the last condition.

So it is clear that

$\mathrm{Hilb}^k(C,0) \subseteq Sp_\gamma$ for all k.

Rmk: (a) If $(C,0)$ is irreducible, then $Sp_\gamma \cong Hilb^N(C,0)$ for N large enough, and in particular Sp_γ is a projective variety.

(b) If $(C,0)$ has r irreducible components, then Sp_γ is an ind-variety with infinitely many irreducible components. There is a lattice action $\mathbb{Z}^{r-1}$ on Sp_γ, and a $(\mathbb{C}^*)^{r-1}$ action.

Even in the non-irreducible case, Sp_γ is in some sense the "$\lim_{k\to\infty}$" $Hilb^k(C,0)$. One formal way of saying this is,

Thm (Maulik–Yun, Migliorini–Shende):

$$\bigoplus_{k=0}^{\infty} H^*(Hilb^k(C,0)) = (gr_p H^*(Sp_\gamma)) \otimes \mathbb{C}[x]$$

(certain perverse filtration)

And recall ORS conj:

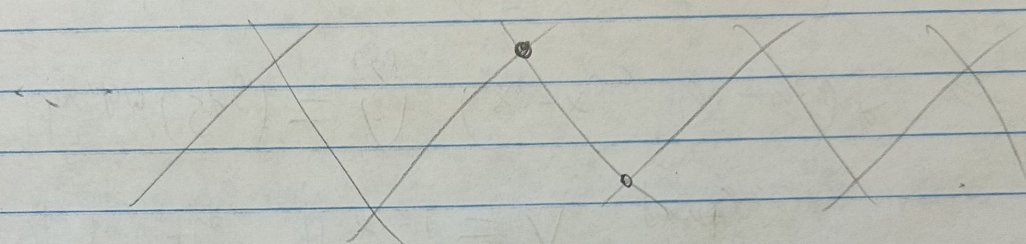
$$HHH^0(L) = \bigoplus_{k=0}^{\infty} H^*(Hilb^k(C,0))$$

* Compactified Jacobian

Ex: $C = \{x^2 = y^2\}$,

$$\gamma = \begin{pmatrix} 0 & x^2 \\ 1 & 0 \end{pmatrix} \sim \begin{pmatrix} x & 0 \\ 0 & -x \end{pmatrix}$$

Then Sp_γ is a chain of $\mathbb{P}^1$'s



Can see lattice ($\mathbb{Z}$) action and $\mathbb{C}^*$ action by stretching

Homology is

$\mathbb{C}[x^{\pm}] \quad 0 \quad \mathbb{C}$

which is analogous to $HHH^0(T^2)$, which was

$$\mathbb{C}[x_1, x_2] \quad 0 \quad \frac{\mathbb{C}[x_1, x_2]}{x_1 - x_2}.$$

This relationship has been demonstrated for torus links by Oscar Kivinen. If $C = \{x^{kn} = y^n\} \leftrightarrow (n, kn)$ torus link, then

$$\gamma = \begin{pmatrix} z_1 x^k & & \\ & \ddots & \\ & & z_n x^k \end{pmatrix}$$

[up to conjugation in SL_n]

where $z_1, \ldots z_n$ are distinct.

$T = (\mathbb{C}^*)^{n-1}$ $\qquad$ $R = \underbrace{\mathbb{C}[x_1^{\pm}, \ldots, x_n^{\pm}]}_{\text{lattice}} / \underbrace{\left(\prod_i x_i - 1\right)}_{SL_n \text{ conditions}}$

Thm: $H_{*,BM}(Sp_\gamma) = HHH^0(T(n,kn)) \otimes_R R$

and

$H^T_{*,BM}(Sp_\gamma) = \underbrace{HY^0(T(n,kn))}_{\text{"y-ification"}} \otimes_R R$

This has been shown by just computing both sides, Up to denominators.

$H^T_{*,BM}(Sp_\gamma) = \bigcap_{i<j} (x_i - x_j, y_i - y_j)^k$ as

a module over $\mathbb{C}[\underbrace{x_1, \ldots, x_n}_{\text{lattice}}, \underbrace{y_1, \ldots, y_n}_{H^*_T(pt)}]$

I extended this to $f = \prod_{i=1}^{n} (z_i x^{d_i} - y)$,

$$\gamma = \begin{pmatrix} z_1 x^{d_1} & & \\ & \ddots & \\ & & z_n x^{d_n} \end{pmatrix},$$

up to denominators,

$$H^T_{*,BM}(S_{p_\gamma}) = \bigcap_{i<j} (x_i - x_j, y_i - y_j)^{d_{ij}}$$

where $d_{ij} = \min(d_i, d_j)$.