Meeting 5: Y-ification

Monday, May 1, 2023 3:10 PM

$\beta \longrightarrow HHH(\beta)$ KhR homology module over $x_1, \dots, x_n$

$\longrightarrow HY(\beta)$ = "y-ificatied homology"

module over $x_1, \dots, x_n$

$y_1, \dots, y_n$

<u>Rouquier complexes</u>

$T_i = [B_i \xrightarrow{b_i} R]$

NOTE: $b_i(1) = 1$

$$R = \frac{\mathbb{C}[x_1, \dots, x_n, x_1', \dots, x_n']}{(x_i = x_i')}$$

$$B_i = \frac{\mathbb{C}[x_1, \dots, x_n, x_1', \dots, x_n']}{\begin{pmatrix} x_i + x_{i+1} = x_i' + x_{i+1}' \quad (*) \\ x_i x_{i+1} = x_i' x_{i+1}' \\ x_j = x_j' \text{ for } j \neq i, i+1 \end{pmatrix}}$$

$T_i^{-1} = [R \xrightarrow{b_i^*} B_i]$

NOTE: $b_i^*(1) = x_i - x_{i+1}'$

<u>Checked</u>: b_i, b_i^* well defined maps of b_i-module

$y = y_1, \dots, y_n$

$$\tilde{T}_i = [B_i[y] \underset{b_i^*(y_i - y_{i+1})}{\overset{b_i}{\rightleftarrows}} R[y]]$$

$D^2 =$ either $b_i b_i^*(y_i - y_{i+1})$

or $b_i^* b_i (y_i - y_{i+1})$ $= (x_i - x_{i+1}')(y_i - y_{i+1})$

$$= \underbrace{(x_i - x_{i+1}')\, y_i + (x_{i+1} - x_i')^{*}\, y_{i+1} + \sum_{j \neq i} (x_j - x_j')\, y_j}_{\sum_{j=1}^{n} (x_j - x'_{\sigma(j)})\, y_j \text{ where } \sigma = (i \ \ i+1)}$$

* used relation $x_i - x_{i+1}' = -(x_{i+1} - x_i')$ $(*)$

* $b_i b_i^* = b_i^* b_i = x_i - x_{i+1}'$

Remark:

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1) we can "unfold" this as a chain complex

$$\begin{tikzcd}
B_i \arrow[r, "b_i"] & R \arrow[dl, "b_i^*"] \\
(y_i - y_{i+1}) B_i \arrow[r, "b_i"] & R(y_i - y_{i+1}) \arrow[dl, "b_i^*"] \\
(y_i - y_{i+1})^2 B_i \arrow[r, "b_i"] & R(y_i - y_{i+1})^2 \arrow[dl, "b_i^*"] \\
(y_i - y_{i+1})^3 B_i \arrow[r, "b_i"] & R(y_i - y_{i+1})^3 \arrow[dl, "b_i^*"] \\
{} &
\end{tikzcd}$$

* Assume y_i have homological degree 2

2) We can work w/ "curved complexes" w/ $D^2 = W$ ← y-ification

provided W is fixed. Can define chain maps, homotopies, ...

ex:
$$\begin{tikzcd}
{} \arrow[r] & C_i \arrow[r, "D_C"] \arrow[d, "f"] & C_{i+1} \arrow[r, "D_C"] \arrow[d, "f"] & C_{i+2} \arrow[r] & \cdots \\
{} \arrow[r] & K_i \arrow[r, "D_K"] & K_{i+1} \arrow[r, "D_K"] & K_{i+2} \arrow[r] & \cdots
\end{tikzcd}$$

$$fD_C = D_K f$$

$$\widetilde{T}_i^{-1} = \left[R[y] \underset{b_i(y_i - y_{i+1})}{\overset{b_i^*}{\rightleftarrows}} B_i[y] \right]$$

Thm: $\widetilde{T}_i \otimes_{R[y]} \widetilde{T}_i^{-1} \simeq R[y]$

$\widetilde{T}_i \otimes \widetilde{T}_{i+1} \otimes \widetilde{T}_i \simeq \widetilde{T}_{i+1} \widetilde{T}_i \widetilde{T}_{i+1}$

All braid relations for $\widetilde{T}_i$ are satisfied up to homotopy

Remark: (C, D_C), $D_C^2 = W_C$ ← complex

(K, D_K), $D_K^2 = W_K$

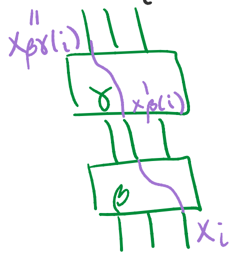
$\longrightarrow$ Can define $(C \otimes K, \underbrace{D_C \otimes 1 \pm 1 \otimes D_K}_{D_{C\otimes K}})$

$$D_{C\otimes K}^2 = W_C + W_K$$

Thm: $\widetilde{T}_\beta$ = product of $\widetilde{T}_i$ has potential $D^2 = W_\beta = \sum_j (x_j - x'_{\beta(j)})\, y_i$

regard β as permutation

NOTE:

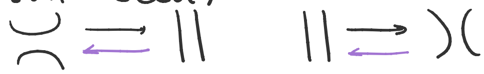


Thm: If we apply HH termwise to $\widetilde{T}_\beta$ & identify the y-variables on same connected component of the link

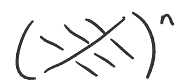
a) $D^2 = 0$

b) The homology is a top. invariant

Facts: 1) Can define this for Khovanov homology (Batson – Seed)



Thm: For $T(n,n) = n,n$ torus link

n unknotted component pairwise linked 

a) $HY(T(n,n)) = \bigcap_{i \neq j} (x_i - x_j,\, y_i - y_j,\, \theta_i - \theta_j)$

as a submodule of $\mathbb{C}[x_1, \dots, x_n, y_1, \dots, y_n, \theta_1, \dots, \theta_n] = HY(\text{unlink})$

b) $HHH(T(n,n)) = \underbrace{\dfrac{HY(T(n,n))}{(y_1, \dots, y_n)\, HY(T(n,n))}}_{\text{easiest description of HHH}}$ } still a module over $\mathbb{C}[x_1, \dots, x_n, \theta_1, \dots, \theta_n]$

Thm: HY(any link) has an action of $sl_2 \Rightarrow$ symmetric

The symmetry exchange x_i & y_i