Y-ification: properties

Recall: $D = d + \sum y_i(\cdots)$

$\uparrow$ original differential

y-ified differential

$$T_i = B_i \underset{b_i^*(y_i - y_{i+1})}{\overset{b_i}{\rightleftarrows}} R$$

$$T_i^{-1} = R \underset{b_i(y_i - y_{i+1})}{\overset{b_i^*}{\rightleftarrows}} B_i$$

Thm 1: If $HHH(\beta)$ is supported in even degrees

Then

1) $HY(\beta)$ is free over $\mathbb{C}[y_1, \ldots, y_n]$

2) $HHH(\beta) = HY(\beta) / (y_1, \ldots, y_r) HY(\beta)$

proof: Spectral sequence

$$\underbrace{HHH(\beta)}_{\text{even}} \otimes \underbrace{\mathbb{C}[y_1, \ldots, y_n]}_{\text{even}} \longrightarrow HY(\beta)$$

$\square$

Remark: d_1, d_2 = differentials

$d_1^2 = d_2^2 = 0$

$d_1 d_2 + d_2 d_1 = 0$

$(d_1 + d_2)^2 = 0$

- first take $H^*(C, d_1)$
- then take $H^*(\text{result}, d_2)$

$\vdots$

$H^*(C, d_1 + d_2)$

Thm2: (Elias-Hogancamp)

$HHH(T(n,n))$ is supported in even homological degrees

proof: Soyeun's lecture from last quarter.

$\rightarrow$ keyword: recursions! $\square$

Cor: Thm 1 + Thm 2

$\Rightarrow HY(T(n,n))$ is a free module over $\mathbb{C}[y_1, \dots, y_n]$

lemma: There is a chain map

$$T_i \xrightarrow{\varphi_i} T_i^{-1}$$

which is a homotopy equivalence if $y_i - y_{i+1}$ is invertible, where $\deg(y_i) = 2$

proof:
$$\begin{tikzcd}
\Big[B_i[y] \arrow[r, shift left, "b_i"] \arrow[dr, "1"'] & R[y] \Big] \arrow[l, shift left, "b_i^*(y_i - y_{i+1})"] \arrow[dl, "y_i - y_{i+1}"] \\
\Big[R[y] \arrow[r, shift left] & B_i[y] \Big] \arrow[l, shift left, "b_i(y_i - y_{i+1})"]
\end{tikzcd}$$

φ_i is a homotopy equivalence if $\mathrm{Cone}(\varphi_i)$ is homotopy equivalent to 0.

$$\mathrm{Cone}(\varphi_i) = \Big[R[y] \underset{y_i - y_{i+1}}{\overset{x_i - x_{i+1}}{\rightleftarrows}} R[y] \Big]$$

Gaussian elimination b/c identity map

If $y_i - y_{i+1}$ is invertible, this is contractible! $\square$

Aside: defining Cone

$$\begin{tikzcd}
\cdots \arrow[r] & C_i \arrow[r, "d_c"] \arrow[d, "f"] & C_{i+1} \arrow[r] \arrow[d, "f"] & C_{i+2} \arrow[r] \arrow[d, "f"] & \cdots \\
\cdots \arrow[r] & K_i \arrow[r, "d_K"] & K_{i+1} \arrow[r] & K_{i+2} \arrow[r] & \cdots
\end{tikzcd}$$

$fd_c = d_K f$
chainmap

$\mathrm{Cone}(f) =$

$$\begin{tikzcd}
C_i \arrow[r, "d_c"] \arrow[dr, "f"] & C_{i+1} \arrow[r, "d_c"] \arrow[dr, "f"] & C_{i+2} \\
\oplus & \oplus & \oplus \\
K_{i-1} \arrow[r, "(-1)^i d_K"'] & K_i \arrow[r, "(-1)^{i+1} d_K"'] & K_{i+1}
\end{tikzcd}$$

$$\begin{tikzcd}
K \arrow[r, hook] & \mathrm{Cone} \arrow[d] \\
 & C
\end{tikzcd}$$

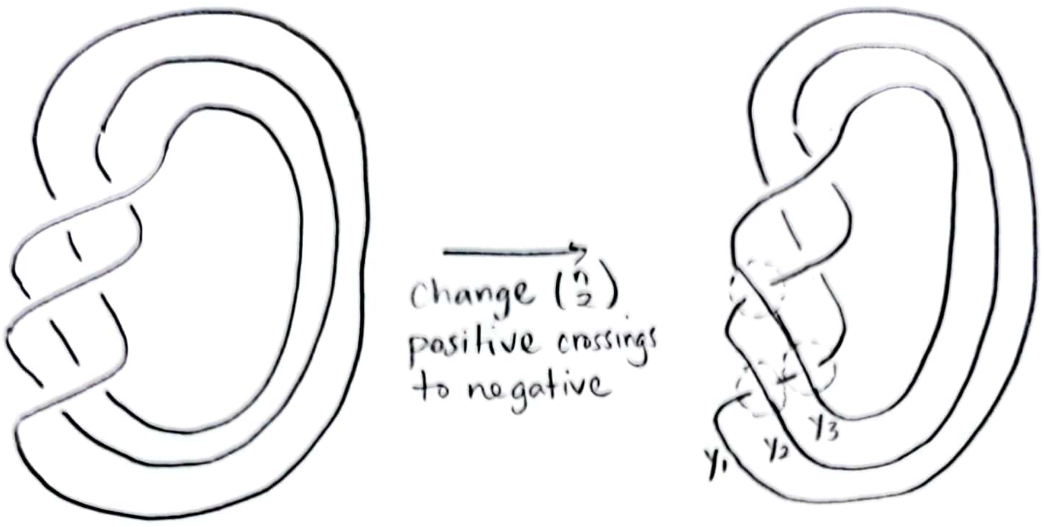
NOTE: This is a categorification of skein relation

$$\underset{T_i}{\times} - \underset{T_i^{-1}}{\times} = (q-q^{-1})\ )($$

Thm: We have a chain map

$$T(n,n) \xrightarrow{\Psi} \text{unlink}$$

homotopy equivalent if <u>all</u> $y_i - y_j$ are invertible



$T(n,n)$ unlink w/ 3 components

Cor: We have a map in HY

$$HY(T(n,n)) \xrightarrow{\Psi} HY(\text{unlink})$$

free over $\mathbb{C}[y_1, \dots, y_n]$

isomorphic if all $y_i - y_j$ are invertible

Cor: Ψ is injective!

proof: $M, N =$ free $\mathbb{C}[y_1, \dots, y_n]$-module

$$M \xrightarrow{\Psi} N, \quad M_{loc} \xrightarrow{\sim} N_{loc} \Rightarrow \Psi \text{ injective } (\ker = 0) \quad \square$$

<u>Ex</u>: $\mathbb{C}[x,y] \supset \underset{I}{\overset{\shortparallel}{(x,y)}}$ free over $\mathbb{C}[y]$

$$I \longrightarrow \mathbb{C}[x,y]$$

localize

$$I[y^{-1}] \xrightarrow{\sim} \mathbb{C}[x,y,y^{-1}]$$

$$\begin{tikzcd} T(n,n) \arrow[rr, "\psi"] \arrow[dr] & & \text{unlink} \\ & T(2,2)_{i,j} \arrow[ur, "\psi_{ij}"'] & \end{tikzcd}$$

i-th
j-th component
linked
rest unlinked

*All injective but not necessarily surjective

$$\mathrm{Im}(\psi) \subset \mathrm{Im}(\psi_{ij}) \quad \forall i,j$$

Thm: a) $\mathrm{Im}\,\psi_{ij} = (x_i - x_j,\ y_i - y_j,\ \theta_i - \theta_j)$
in $HY(\text{unlink}) = \mathbb{C}[x_1,\dots,x_n,y_1,\dots,y_n,\theta_1,\dots,\theta_n]$
[computation for $T(2,2)$]

Hartog's lemma

b) $HY(T(n,n)) \simeq \mathrm{Im}\,\psi \overset{!}{\simeq} \bigcap_{i\neq j} \mathrm{Im}\,\psi_{ij}$ ← coincide up to codim $= 2$

$$\bigcap_{i\neq j} \mathrm{Im}\,\psi_{ij} = \bigcap_{i\neq j} (x_i - x_j,\ y_i - y_j,\ \theta_i - \theta_j)$$

proof: (a) $T(2,2) = T^2 \simeq$

$$\begin{tikzcd} B \arrow[r, shift left, "x_1 - x_2'"] \arrow[dr, "b"'] & B \arrow[l, shift left, "y_1 - y_2"] \arrow[r, "b"] & R \arrow[dl, "y_1 - y_2"] \\ & R & \end{tikzcd}$$

Ψ:

unlink

Apply HH_0:

$$\begin{tikzcd} R[y] \arrow[r, shift left, "0"] \arrow[dr, "x_1 - x_2"'] & R[y] \arrow[l, shift left, "y_1 - y_2"] \arrow[r, "x_1 - x_2"] & R[y] \arrow[dl, "y_1 - y_2"] \\ & R & \end{tikzcd}$$

z $\quad \alpha \quad$ w

$$HH_0(T(2,2)) = \frac{R[y]\langle z,w\rangle}{w(x_1-x_2) = z(y_1-y_2)}$$

$$d(\alpha) = (x_1-x_2)w \pm (y_1-y_2)z \; \big\} \text{ im}$$

$$\left.\begin{aligned} d(w) &= 0 \\ d(z) &= 0 \end{aligned}\right\} \text{ generates ker}$$

$$\Psi(\alpha) = 0$$

$$\Psi(w) = y_1 - y_2$$

$$\Psi(z) = x_1 - x_2$$

$\longrightarrow \Psi$ injective, image $= (x_1-x_2, y_1-y_2)$