# Meeting 2: Braid Varieties

Monday, April 10, 2023 3:08 PM

## Last time:

$w = s_{i_1}, \dots, s_{i_k}$ braidword

$$\mathcal{F}^{(0)} \overset{s_{i_1}}{\text{——}} \mathcal{F}^{(1)} \overset{s_{i_2}}{\text{——}} \mathcal{F}^{(2)} \quad \cdots \overset{s_{i_k}}{\text{——}} \mathcal{F}^{(k)}$$

flags

sequence of flags

Conditions:

1) "Open Bott-Samelson"

$$\begin{array}{ccc}
\mathcal{F}^{(j)} & & \mathcal{F}^{(j+1)} \\
\| & & \| \\
\mathcal{F}_n^{(j)} & = & \mathcal{F}_n^{(j+1)} \\
\cup & & \cup \\
\mathcal{F}_{n-1}^{(j)} & = & \mathcal{F}_{n-1}^{(j+1)} \\
\cup & & \cup \\
\vdots & & \vdots \\
\cup & & \cup \\
\mathcal{F}_{i_j}^{(j)} & \neq & \mathcal{F}_{i_j}^{(j+1)} \\
\cup & & \cup \\
\vdots & & \vdots \\
\cup & & \cup \\
\mathcal{F}_0^{(j)} & = & \mathcal{F}_0^{(j+1)}
\end{array}$$

where $\mathcal{F}^{(j)}$ & $\mathcal{F}^{(j+1)}$ are in position $s_{i_j}$

$\Updownarrow$

$\mathcal{F}_k^{(j)} = \mathcal{F}_k^{(j+1)}$, $\mathcal{F}_{i_j}^{(j)} \neq \mathcal{F}_{i_j}^{(j+1)}$

where $k \neq i_j$

ex: $n=3$

$$\underbrace{\begin{array}{ccc}
\mathbb{C}^3 & & \mathbb{C}^3 \\
\cup & & \cup \\
P & = & P' \\
\cup & & \cup \\
L & \neq & L' \\
\cup & & \cup \\
0 & & 0
\end{array}}_{s_1}
\qquad\qquad
\underbrace{\begin{array}{ccc}
\mathbb{C}^3 & & \mathbb{C}^3 \\
\cup & & \cup \\
P & \neq & P' \\
\cup & & \cup \\
L & = & L' \\
\cup & & \cup \\
0 & & 0
\end{array}}_{s_2}$$

2) "brick mfld" conditions

$\mathcal{F}^{(0)}$ = standard flag

$\mathcal{F}^{(k)} = w_0 \mathcal{F}^{(0)}$ where $w_0 = \big( 0 \quad 1 \big)$

$\mathcal{F}^{(0)}$ = standard flag
$\mathcal{F}^{(k)} = w_0 \mathcal{F}^{(0)}$ where $w_0 = \begin{pmatrix} 0 & & 1 \\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & 0 \end{pmatrix}$

Thm: (last time) This is smooth (or empty)
$\dim = \ell(w) - \binom{n}{2}$

Ex: $n=2$

$w = s_1^3$

$\{ \underset{e_1}{\overset{\shortparallel}{L_1}} \neq L_2 \neq L_3 \neq \underset{e_2}{\overset{\shortparallel}{L_4}} \} \subset \mathbb{P}^1_{L_2} \times \mathbb{P}^1_{L_3}$

Case 1: $L_1 = L_3$

$\Rightarrow$ no choice for $L_3$

$L_2 \neq L_1 \Rightarrow \underbrace{\mathbb{C}P^1 - pt}_{\cong \mathbb{C}}$ choices for $L_2$

$\Rightarrow \mathbb{C}$

Case 2: $L_1 \neq L_3$

- $L_3 \neq L_1, L_4 \Rightarrow \mathbb{C}P^1 - 2pts = \mathbb{C}^*$ choices for $L_3$
- Once $L_3$ choosen

$L_2 \neq L_1, L_3 \Rightarrow \mathbb{C}P^1 - 2\,pts = \mathbb{C}^*$ choices for $L_2$

$(\mathbb{C}^* \times \mathbb{C}^*)$

Conclusion:

$X(\sigma^3)$ has a stratification $X(\sigma^3) = \mathbb{C} \sqcup (\mathbb{C}^*)^2$

Cor: # points over $\mathbb{F}_q$ is $q + (q-1)^2 = q^2 - q + 1$

Thm: a) $w = \cdots s_i s_{i+1} s_i \cdots$

$w' = \cdots s_{i+1} s_i s_{i+1} \cdots$

Then $X(w) \simeq X(w')$ (last time)

b) $w = \cdots s_i s_i \cdots$

$w' = \cdots s_i \cdots$

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$w' = \cdots s_i \cdots$

$w'' = \cdots 1 \cdots$

$X(w)$ has a stratification

$$X(w) \simeq X(w') \times \mathbb{C}^* \sqcup X(w'') \times \mathbb{C}$$

proof:

$$\cdots \mathcal{F}^{(1)} \overset{s_i}{\text{---}} \mathcal{F}^{(2)} \overset{s_i}{\text{---}} \mathcal{F}^{(3)} \cdots$$

$\mathcal{F}^{(1)} = \mathcal{F}^{(2)} = \mathcal{F}^{(3)}$ same except $i^{th}$ position

Case 1: $\mathcal{F}^{(1)} = \mathcal{F}^{(3)}$

$\underset{\overset{\shortparallel}{\mathbb{C}}}{(\mathbb{P}^1 - pt)}$ choices for $\mathcal{F}^{(2)}$

Case 2: $\mathcal{F}^{(1)} \neq \mathcal{F}^{(3)}$

$(\mathbb{P}^1 - 2pt) = \mathbb{C}^*$ choices for $\mathcal{F}^{(2)}$ $\square$

$$\# X(w) = \# X(w')(q-1) + \# X(w'') \cdot q$$

$\Rightarrow$ Can compute $\# X(w)$ recursively for any $w$!

Thm: (Kalman)

$X(w) = (a=0)$ term in HOMFLY $(\underset{\text{positive braid}}{w} \cdot \underset{\text{negative braid}}{\Delta^{-1}})$

where $\Delta = w_0$

def: $B_i(z) = \begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & z & -1 & \\ & & 1 & 0 & \\ & & & & \ddots \\ & & & & & 1 \end{pmatrix}$ $n \times n$ matrix

$\qquad\qquad\qquad\quad i \quad i+1$

lemma: $B_i(z_1) B_{i+1}(z_2) B_i(z_3) = B_{i+1}(z_3) B_i(z_1 z_3 - z_2) B_{i+1}(z_1)$

proof: $B_i(z_1)\,B_{i+1}(z_2)\,B_i(z_3) = \begin{pmatrix} z_1 & -1 & \\ 1 & 0 & \\ & & 1 \end{pmatrix}\begin{pmatrix} 1 & & \\ & z_2 & -1 \\ & 1 & 0 \end{pmatrix}\begin{pmatrix} z_3 & -1 & \\ 1 & 0 & \\ & & 1 \end{pmatrix}$

$$= \begin{pmatrix} z_1 & -z_2 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} z_3 & -1 & \\ 1 & 0 & \\ & & 1 \end{pmatrix} = \begin{pmatrix} z_1z_3 - z_2 & z_1 & 1 \\ z_3 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

Then compute $B_{i+1}(z_3)\,B_i(z_1z_3 - z_2)\,B_{i+1}(z_1)$ $\square$

$w = s_{i_1} \cdots s_{i_k} \rightsquigarrow$ Braid matrix $B_{i_1}(z_1) \cdots B_{i_k}(z_k)$

invariant under braid moves up to change of variables

Thm: $X(w) \simeq \{ z_1 \cdots z_k : B_{i_1}(z_1) \cdots B_{i_k}(z_k) = w_0 U \}$

$U$ is upper triangular

$\iff \{ z_1 \cdots z_k : w_0^{-1} B_{i_1}(z_1) \cdots B_{i_k}(z_k)$ upper triangular$\}$

<u>Ex</u>: $w = s_1^3$

$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} z_1 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} z_2 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} z_3 & -1 \\ 1 & 0 \end{pmatrix}$ upper triangular

$$= \begin{pmatrix} 1 & 0 \\ z_1 & -1 \end{pmatrix}\begin{pmatrix} z_2z_3 - 1 & -z_2 \\ z_3 & -1 \end{pmatrix} = \begin{pmatrix} z_2z_3 - 1 & -z_2 \\ \boxed{z_1z_2z_3 - z_1 - z_3} & 1 - z_1z_2 \end{pmatrix}$$

upper triangular when $= 0$

$X(s^3) = \{ z_1, z_2, z_3 : z_1z_2z_3 - z_1 - z_3 = 0 \} \underset{\text{closed}}{\subset} \mathbb{C}^3$

$\simeq \{ z_1z_2 - 1 \neq 0 \} \underset{\text{open}}{\subset} \mathbb{C}^2$ smooth

$z_1 = z_3(z_1z_2 - 1)$

Case 1: $z_1z_2 - 1 \neq 0$

$z_3 = \dfrac{z_1}{z_1z_2 - 1}$

Case 2: $z_1z_2 - 1 = 0$

$z_1 = 0$ contradiction!

proof of theorem: want to find bijection btwn flags

$$\mathcal{F} \xrightarrow{s_i} \mathcal{F}'$$

lemma: $\mathcal{F}, \mathcal{F}'$ are in position $s_i$ if $\mathcal{F}' = \mathcal{F} B_i(z)$

Unpack: Choose some basis compatible w/ $\mathcal{F}$

$v_1, v_2, \ldots, v_n$

$\mathcal{F}_0 = 0 \subset \mathcal{F}_1 = \langle v_1 \rangle \subset \mathcal{F}_2 = \langle v_1, v_2 \rangle \subset \ldots$

↳ non unique, make a choice

$$\mathcal{F} \sim \begin{pmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{pmatrix} \qquad \begin{pmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{pmatrix} \cdot B_i(z) \sim \mathcal{F}'$$

ex: $$\begin{pmatrix} | & | \\ v_1 & v_2 \\ | & | \end{pmatrix} = \begin{pmatrix} | & | \\ zv_1 + v_2 & -v_1 \\ | & | \end{pmatrix}$$

any line transverse to span$(v_1)$ has unique generator of this form