Meeting 4

Monday, April 24, 2023 3:12 PM

Goal: $H_*(X(\beta))$ vs. HHH

X = alg. variety (quasi-projective)

Fact: (Deligne) $H_*(X)$ has a weight filtration

$$\cdots \subset W_{j+1,i} \subset W_{j,i} \subset H_i(X)$$

In each $H_i(X)$ we have a sequence of subspaces

$$W_{j-1} \subset W_j \subset W_{j+1} \subset W_{j+2} \subset \cdots$$

Associated graded:

$$gr_j^W H_i = W_{j,i} / W_{j-1,i}$$

Thm: If X is smooth & projective, then

$$W_{j,i} = \begin{cases} H_i & j \geq i \\ 0 & j < i \end{cases}$$

$$0 \subset \cdots \subset 0 \subset H_i(x) \subset H_i(x) \subset \cdots$$

def: X = smooth projective

$\cup$

$D = \bigcup_{i=1}^{k} D_i$ is a (strict) normal crossing divisor

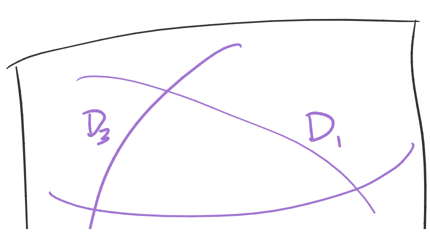
if we have the following:

a) $\dim D_i = \dim X - 1$, D_i are smooth

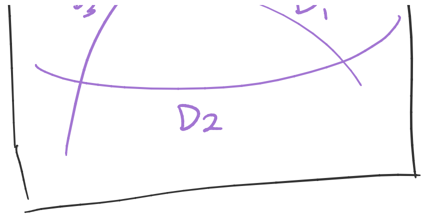
b) for all $I \subset \{1, \ldots, k\}$ define

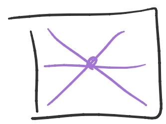
$D_I = \bigcap_{j \in I} D_j$ either empty or smooth & connected, reduced of dimension $\dim X - |I|$

D_j $j \in I$ intersect transversely



X



$\longrightarrow$ codim fails if 

Thm 2: $X \supset D = \bigcup_{i=1}^{k} D_i$, $\quad Y = X \setminus D$ open in X

smooth, projective

Normal crossing divisor

$$\begin{tikzcd}[column sep=small]
 & & H_*(D_1) \arrow[dr] & \\
\cdots \bigoplus H_*(D_i \cap D_j \cap D_k) \arrow[r] & \bigoplus H_*(D_i \cap D_j) \arrow[ur] \arrow[dr] & \oplus \; \vdots \; \oplus & H_*(X) \\
 & & H_*(D_k) \arrow[ur] &
\end{tikzcd}$$

Complex, terms $= \bigoplus_{I \subset \{1,\dots,k\}} H_*(D_I)$, differential given by inclusion (up to sign)

- $d^2 = 0$
- Homology of the complex $= gr^W H_*(Y)$

  Bigraded by $|I|$ + homological degree $= W$, homological

Fact from spectral seq. collapsing

Ex: 1) $k=2$

$$\begin{tikzcd}
 & H_*(D_1) \arrow[dr] & \\
H_*(D_1 \cap D_2) \arrow[ur] \arrow[dr, "-"'] & & H_*(X) \\
 & H_*(D_2) \arrow[ur] &
\end{tikzcd}$$

2) $X = \mathbb{C}P^1$

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$D_1 = \{0\}$, $D_2 = \infty$ $\Big\}$ codim $= 1$

$Y = \mathbb{C}P^1 \setminus \{0, \infty\} = \mathbb{C}^*$

$\longrightarrow$ would be 3 different complexes

$$\begin{tikzcd} \underset{\mathbb{Z}}{\overset{\|}{H_0(D_1)}} \arrow[dr, "1"] & \\ & H_0(\mathbb{C}P^1) = \mathbb{Z} \\ \underset{\mathbb{Z}}{\overset{\|}{H_0(D_2)}} \arrow[ur, "1"] & \end{tikzcd} \qquad \begin{tikzcd} \underset{0}{\overset{\|}{H_2(D_1)}} \arrow[dr] & \\ & \underset{\mathbb{Z}}{\overset{\|}{H_2(\mathbb{C}P^1)}} \\ \underset{0}{\overset{\|}{H_2(D_2)}} \arrow[ur] & \end{tikzcd}$$

$H_1 = \mathbb{Z}$ $\qquad H_0 = 0$

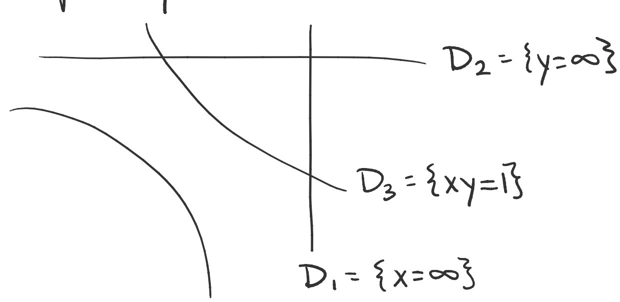
$$\longrightarrow H_0(\mathbb{C}^*) = H_1(\mathbb{C}^*) = \mathbb{Z}$$
$$H_2^{BM}(\mathbb{C}^*) = H_1^{BM}(\mathbb{C}^*) = \mathbb{Z}$$

interesting weight filtration

3) $\{xy - \overset{Y}{\overset{\|}{1}} \neq 0\} \subset \mathbb{C}^2$

Braid variety for $s_1^3 = \beta$

Compactify: $X = \mathbb{P}^1 \times \mathbb{P}^1$



$D = D_1 \cup D_2 \cup D_3$

$Y = X \setminus D$

Claim: $D_1 \simeq D_2 \simeq D_3 \simeq \mathbb{P}^1$

$D_1 \cap D_2 \simeq D_1 \cap D_3 \simeq D_2 \cap D_3 \simeq pt$

$D_1 \cap D_2 \cap D_3 = \emptyset$

$$\begin{tikzcd}
H_*(D_1\cap D_2) \arrow[r] \arrow[dr] & H_*(D_1) \arrow[dr] & \\
H_*(D_1\cap D_3) \arrow[ur] \arrow[dr] & H_*(D_2) \arrow[r] & \underbrace{H_*(\mathbb{P}^1\times\mathbb{P}^1)}_{*=0,2,4} \\
H_*(D_2\cap D_3) \arrow[r] \arrow[ur] & H_*(D_3) \arrow[ur] &
\end{tikzcd}$$

$*=0$ clear $\mathbb{Z}$

$*=4$ clear $\mathbb{Z}$

$*=2$

$$\begin{tikzcd}
0 & \mathbb{Z} \arrow[dr, "{(0,1)}"] & \\
0 & \mathbb{Z} \arrow[r, "{(1,0)}"] & \mathbb{Z}\oplus\mathbb{Z} \\
0 & \mathbb{Z} \arrow[ur, "{(1,1)}"'] &
\end{tikzcd}$$

$\mathbb{Z}$

Observation: Nonempty intersections $\Longleftrightarrow$ subwords of $s_1^3 = s_1 s_1 s_1$ containing s_1

Recall:

$X(\beta) \subset \mathrm{Brick}(\beta)$ smooth projection

Complement $= \bigcup \mathrm{Brick}(\beta')$ — D_i = normal crossing divisor

where $\beta' = \beta$ w/one letter removed

Idea 2: Equivariant homology of Bott-Samelson varieties $\longleftrightarrow$ Bott-Samelson bimodules

$R = \mathbb{C}[x_1, \dots, x_n]$

$B_i = \mathbb{C}[x_1, \dots, x_n, x_1', \dots, x_n'] \Big/ \big(x_i + x_{i+1} = x_i' + x_{i+1}' \big)$

$$B_i = \frac{\mathbb{C}[x_1,\dots,x_n,x_1',\dots,x_n']}{\left(\begin{array}{l} x_i + x_{i+1} = x_i' + x_{i+1}', \\ x_i x_{i+1} = x_i' x_{i+1}' \\ x_j = x_j', \quad j \neq i, i+1 \end{array}\right)}$$

BS. bimodule:

$$B_{i_1} \otimes_R B_{i_2} \otimes_R \cdots \otimes_R B_{i_k}$$

$$\mathcal{F}_0 \underset{\substack{\text{agree} \\ \text{except } i_1}}{\vee} \mathcal{F}_1 \underset{s_{i_2}}{\vee} \mathcal{F}_2 \underset{s_{i_3}}{\vee} \cdots \underset{s_{i_k}}{\vee} \mathcal{F}_k$$

Recall: Rouquier complexes

$T_i = [B_i \to R]$

$T_{i_1} \otimes T_{i_2} \otimes \cdots \otimes T_{i_k} =$

$$B_{i_1} \otimes \cdots \otimes B_{i_k} \nearrow \begin{array}{c} B_{i_1} \cdots \hat{B}_{i_j} \cdots B_{i_k} \\ \oplus \\ \vdots \end{array} \longrightarrow \quad \cdots \longrightarrow R$$

(skip 1) (skip 2) ...

Q: How do relations appear from line bundles?

$$\begin{array}{ccc} \vdots & & \vdots \\ \mathcal{F}_{i+1} & = & \mathcal{F}_{i+1}' \\ \mathcal{F}_i & ? & \mathcal{F}_i' \\ \mathcal{F}_{i-1} & = & \mathcal{F}_{i-1}' \\ \vdots & & \vdots \\ \mathcal{F}_1 & & \mathcal{F}_1' \\ \mathcal{F}_0 & & \mathcal{F}_0' \end{array}$$

line bundles:

$\mathcal{L}_i = \mathcal{F}_i / \mathcal{F}_{i-1}$ $\mathcal{L}_i' = \mathcal{F}_i' / \mathcal{F}_{i-1}'$

vector bundles

NOTE: • $\mathcal{L}_j = \mathcal{L}_j'$ if $j \neq i, i+1$

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• Rank 2 vector bundle

$$V = \mathcal{F}_{i+1}/\mathcal{F}_{i-1} = \mathcal{F}_{i+1}'/\mathcal{F}_{i-1}' = V'$$

$$\mathcal{L}_i \underset{\text{in } V}{\subset} \text{ line}$$

$$\mathcal{L}_i' = \mathcal{F}_i'/\mathcal{F}_{i-1}' \qquad \mathcal{L}_i' \text{ line in } V' = V$$

$$V'/\mathcal{L}_i' = \mathcal{L}_{i+1}'$$

Chern Classes

X = top. space

V = rank r complex vector bundle

Define classes:

$$c_1(V) \in H^2(X)$$
$$c_2(V) \in H^4(X)$$
$$\vdots$$
$$c_r(V) \in H^{2r}(X)$$

Fact: These are uniquely determined by the following conditions:

1) Functoriality:

$$\begin{array}{ccc} f^*V & & V \\ \downarrow & & \downarrow \\ f: X & \longrightarrow & Y \end{array} \qquad c_i(f^*V) = f^*c_i(V)$$

2) Whitney sum formula:

$K^k \subset V^r$ subbundle $\quad L = V/K^{(r-k)}$

$$(1 + c_1(K) + c_2(K) + \dots + c_k(K))(1 + c_1(L) + c_2(L) + \dots + c_{r-k}(L))$$
$$= (1 + c_1(V) + c_2(V) + \dots + c_r(V))$$

$*$ $c_1(V) = c_1(K) + c_1(L)$

$c_2(V) = c_2(K) + c_2(L) + c_1(K)c_1(L)$

$$\to \quad (1+c_1(\mathcal{L}_i))(1+c_1(\mathcal{L}_{i+1})) = 1+c_1(V)+c_2(V)$$
$$= (1+c_1(\mathcal{L}_i'))(1+c_1(\mathcal{L}_{i+1}'))$$

3) V = rank r vector bundle

X smooth

Choose a section $s: X \longrightarrow V$, $Z = \{s=0\}$ ($\dim Z = \dim X - 2r$)

Poincaré Dual of eq. class of Z

$c_r(V) = PD[Z]$