Meeting 1: Flag, Bott-Samelson, Braid Varieties

Monday, April 3, 2023 3:07 PM

Goals for quarter:

- braid varieties
- y-ification
- $\mathrm{Hilb}(\mathbb{C}^2)$

1) Flag varieties

$\mathbb{C}^n$

$\mathcal{F}\ell_n = \{0 \subset F_1 \subset F_2 \subset \dots \subset F_n\}$

F_i = vector subspace of $\mathbb{C}^n$

$\dim F_i = i$

$e_1, \dots, e_n$ = basis in $\mathbb{C}^n$

$0 \subset \langle e_1 \rangle \subset \langle e_1, e_2 \rangle \subset \langle e_1, e_2, e_3 \rangle \dots = \mathcal{F}_{std}$

"standard flag" = point in $\mathcal{F}\ell_n$

2) Bott-Samelson variety

Input: (positive) braid word

$w = s_{i_1} s_{i_2} \cdots s_{i_k}$

Output: $BS(w) = \{\mathcal{F}^{(0)}, \mathcal{F}^{(1)}, \mathcal{F}^{(2)}, \dots, \mathcal{F}^{(k)}\}$

Each $\mathcal{F}^{(i)}$ is a flag!

e.g. $\mathcal{F}^{(0)} = \{0 \subset F_1^{(0)} \subset F_2^{(0)} \subset \dots \subset F_n^{(0)}\}$

NOTE: $\mathcal{F}^{(s)}$ & $\mathcal{F}^{(s+1)}$ agree except for $F_{i_s}^{(s)}$

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Ex: 1) $n=2$

$\mathcal{F}\ell_2 = \mathbb{P}^1$ = choice of a line

$w = s_1^K$

$$\mathcal{BS}(s_1^K) = \underbrace{\mathbb{P}^1 \times \mathbb{P}^1 \times \cdots \times \mathbb{P}^1}_{K} \quad \text{no conditions}$$

2) $n=3$

$w = s_1 s_2 s_2 s_1$

$$\begin{array}{lccccccccc}
 & \mathbb{C}^3 & & & & & & & & \\
 & \cup & & & & & & & & \\
\text{plane} & P_0 & = & P_0 & s_2 & P_2 & s_2 & P_3 & = & P_3 \\
 & \cup & & \cup & & & & \cup & & \cup \\
\text{line} & L_0 & \xleftrightarrow[s_1]{\text{potentially different}} & L_1 & = & L_1 & = & L_1 & s_1 & L_4 \\
 & \cup & & \cup & & \cup & & \cup & & \cup \\
 & 0 & & 0 & & 0 & & 0 & & 0 \\
 & \mathcal{F}^{(0)} & & \mathcal{F}^{(1)} & & \mathcal{F}^{(2)} & & \mathcal{F}^{(3)} & & \mathcal{F}^{(4)}
\end{array}$$

$$\begin{array}{ccccc}
P_0 & & P_2 & P_3 & \supset \\
\cup & \searrow\!\!\supset & \cup & \subset\!\!\nearrow & \\
L_0 & & L_1 & & L_4
\end{array}$$

Fact: 1) <u>Does</u> depend on the order of s_i
& choice of braid word

& choice of braid word

2) $BS(ws_i) \longrightarrow BS(w)$ ($\Rightarrow$ smooth)

forget last flag

$\mathbb{P}^1$ - fibration

line in F_{i+1}/F_{i-1}

e.g. $\underbrace{F_{i-1} \subset F_i \subset F_{i+1}}^{F_i}$

fixed fixed

<u>Modifications</u>

I) Brick manifold (L. Escobar)

<u>Brick(w)</u> Same, $F^{(0)}$ = std. coordinate flag

$F^{(last)} = w_0 F^{(0)}$

where $w_0 = [n \;\; n-1 \;\; n-2 \; \cdots \; 1]$

Thm: (Escobar) Brick(w) is either empty or smooth of dimension $\ell(w) - \binom{n}{2}$

$w \supset$ reduced subexpression for w_0

$\Updownarrow$

$\delta(w) = w_0$

Demazure product

II) Open Bott-Samelson

$P_0 = P_0 \neq P_2 \neq P_3 = P_3$

$L_0 \neq L_1 = L_1 = L_1 \neq L_4$

$\mathcal{F}^{(0)}$ & $\mathcal{F}^{(1)}$ are in position s_i

if $\mathcal{F}_j^{(0)} = \mathcal{F}_j^{(1)} \quad j \neq i \leftarrow$ BS

$\tilde{\mathcal{F}}_i^{(0)} = \tilde{\mathcal{F}}_i^{(1)} \leftarrow$ New

$$\mathring{BS}(w) \underset{open}{\subset} BS(w)$$

Fact: $\mathring{BS}(w)$ does not depend on braid word

$$w = \cdots s_i s_{i+1} s_i \cdots \longleftrightarrow \cdots s_{i+1} s_i s_{i+1} \cdots$$

<u>Idea</u>: $w = s_1 s_2 s_1$

$P_0 \qquad P_0 \neq P_1 \qquad \subset \quad P_1 \qquad P_2$ but only necessary to be different from P_0

$L_0 \neq L_1 \qquad L_1 \neq L_3$

Know: P_0, L_0

P_1, L_3 = "boundary"

$$\boxed{L_1 = P_0 \cap P_1}$$

$BS(s_1 s_2 s_2 s_1)$

$BS(s_1 s_2 s_2 s_1) \quad \subset \quad BS(s_1 s_2 s_2 s_1)$

$BS(s_1 s_2 s_2 s_1)$

$BS(s_1 s_2 s_2 s_1)$

purple = remove

$$BS(\text{any subword of } w) \subset BS(w)$$

$\mathring{BS}(w)$ = complement to the union of these

↪ normal crossing divisor w/ nice properties

III) Open brick manifold = braid variety

$$= \underbrace{\mathring{BS}(w)}_{\text{flag differ at every step}} \cap \underbrace{\mathrm{Brick}(w)}_{\text{fix first \& last}} = X(w)$$

Conclusion: $X(w)$ is smooth & independent of braid word

<u>Ex</u>: $n=2$

$w = s_1 s_1 \cdots s_1$

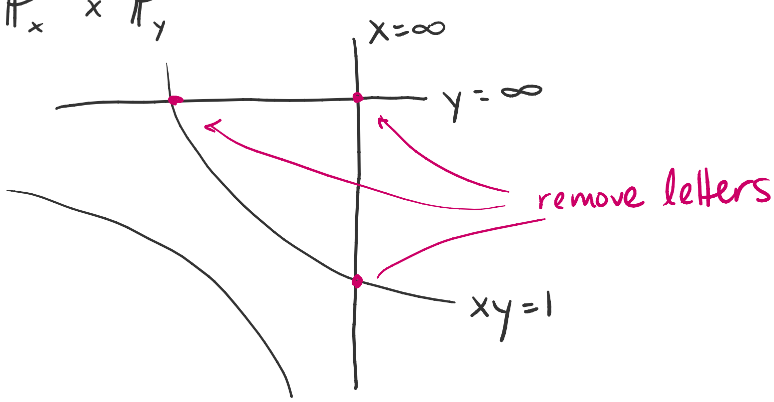
$$\underset{\text{fix}}{L_0} \neq L_1 \neq L_2 \neq \cdots \neq \underset{\text{fix}}{L_k}$$

$\mathrm{Brick}(w) = (\mathbb{P}^1)^{k-1}$

$w = s_1 s_1 s_1$

$$\underset{\text{fix}}{L_0} \neq L_1 \neq L_2 \neq \underset{\text{fix}}{L_3}$$

$\mathbb{P}^1_x \times \mathbb{P}^1_y$



$\mathrm{Brick}_3(s_1^2) < \mathrm{Brick}(s_1^3)$

$\{x=\infty\}$ $\{y=\infty\}$ $\{xy=1\}$

Remove $\boxed{\{xy\neq 1\}\subset \mathbb{C}^2}$