# Lasagna! - 0-Framed unknot

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[Part 1]: Calculate $S_0^2(S^2\times D^2;\emptyset) \cong \underline{Kh}_2(U^0)$.

• Note: $S_0^2(S^2\times D^2;\emptyset)$ has a natural ring structure given by multiplication map:

$$m: S_0^2(S^2\times D^2;\emptyset)\otimes S_0^2(S^2\times D^2;\emptyset) \longrightarrow S_0^2(S^2\times D^2;\emptyset)$$
$$m(\mathcal{F}_1\otimes\mathcal{F}_2) = \mathcal{F}_1\cup\mathcal{F}_2$$

... coming from gluing $S^2\times D^2$'s together along $S^2\times I$, $I\subset\partial D^2$.

⇝ We aim to prove the following:

• Thm: $S^2_{0,0,*}(S^2\times D^2;\emptyset) \cong \mathbb{K}\langle A_1, A_0, A_0^{-1}\rangle$, $\deg_q(A_i) = -2i$.

Proof.

⇝ To do this, we need to describe $\beta$ & $\psi^{[m]}$ explicitly on cables of $U^0$ (0-framed unknot). We use the following lemma for $\beta$:

• Lemma (Grigsby - Licata - Wehrli): The braid group action on the cable of a knot factors through $S_n$.

• Note: $\underline{Kh}_{2,\alpha}(U^0) = \left(\bigoplus_{r\in\mathbb{N}} q^{-2r-|\alpha|}\, Kh(U_0(r-\alpha^-, r+\alpha^+))\right)/\sim$

... where $U_0(r-\alpha^-, r+\alpha^+)$ is just a $2r+|\alpha|$ component unlink. Thus, letting $A = \mathbb{K}[X]/\langle X^2\rangle\{-1\}$, we have:

$$Kh_2(U^0(r-\alpha^-, r+\alpha^+)) \cong A^{\otimes(2r+|\alpha|)}$$

... We may then write:

$$\underline{Kh}_{2,\alpha}(U^0) = \left(\bigoplus_{r\in\mathbb{N}} q^{-1}A^{\otimes(r-\alpha^-)}\otimes A^{\otimes(r+\alpha^+)}\right)/\sim$$

⇝ By our lemma, we can re-write the above as:

$$\underline{Kh}_{2,\alpha}(U^0) = \left(\bigoplus_{r\in\mathbb{N}} \mathrm{Sym}^{r-\alpha^-}(q^{-1}A)\otimes \mathrm{Sym}^{r+\alpha^+}(q^{-1}A)\right)/\sim'$$

... where $\sim'$: $\psi^{[0]}(v)\sim 0$, $\psi^{[1]}(v)\sim v$.

• Observation: $\psi^{[m]}$ are induced by splits, and therefore we can write them as:

$$\psi^{[1]}(v) = \Delta(1)\cdot v$$
$$\psi^{[0]}(v) = \Delta(X)\cdot v$$

... where '$\cdot$' is the multiplication from the ring structure on $S_0^2(S^2\times D^2;\emptyset)$.

⇝ Let $A_0 = 1$ and $A_1 = X$, then we can write our cabled Khovanov homology summands as:

$$\mathrm{Sym}^*(q^{-1}A) \cong \mathbb{K}\langle A_0, A_1\rangle$$

... letting the second tensor factor be $\mathbb{K}\langle \overset{1}{B_0}, \overset{X}{B_1}\rangle$ gives:

$$\mathrm{Sym}^*(q^{-1}A)\otimes \mathrm{Sym}^*(q^{-1}A) \cong \mathbb{K}\langle A_0, A_1, B_0, B_1\rangle$$

⇝ Furthermore, we can write $\Delta(1) = A_0B_1 + A_1B_0$ and $\Delta(X) = A_1B_1$, letting $I$ be the ideal generated by $\{A_0B_1 + A_1B_0,\ A_1B_1 - 1\}$, we have:

$$\underline{Kh}_2(U^0) \cong \mathrm{Sym}^*(q^{-1}A)\otimes\mathrm{Sym}^*(q^{-1}A)/\sim' \cong \mathbb{K}\langle A_0, A_1, B_0, B_1\rangle/I$$

... the second relation implies $B_1 = A_1^{-1}$, and using the first we obtain: $B_0 = -A_1^{-1}A_0A_1^{-1}$. Thus:

$$\underline{Kh}_2(U^0) \cong \mathbb{K}\langle A_0, A_1, A_1^{-1}\rangle,$$ as desired. ∎