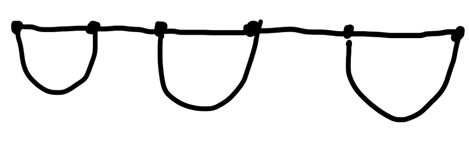
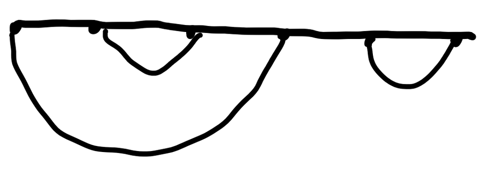
# Lasagna! - Arc Algebras

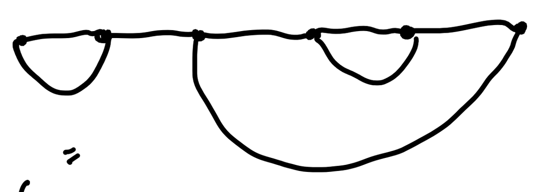
Tuesday, March 14, 2023 1:34 PM

• **Note:** Khovanov extended his link invariant to tangles, producing a tangle invariant valued in chain complexes of $H_n$-bimodules. This POV will be very useful for the calculation of $S_0^2$.

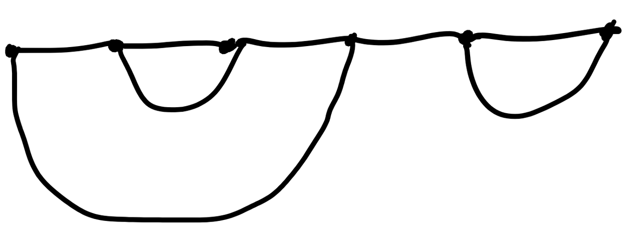
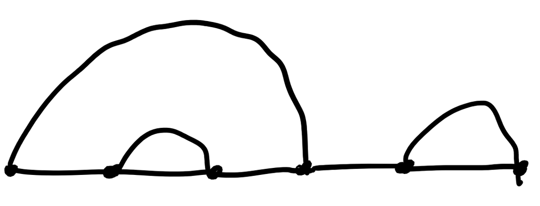
**(Def)** $\mathcal{C}_n$ := the set of crossingless matchings on $2n$ points.

**Ex.** $\mathcal{C}_3$ :

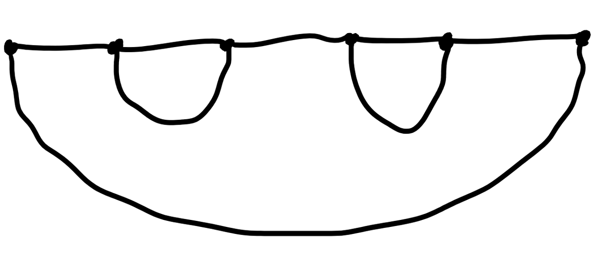
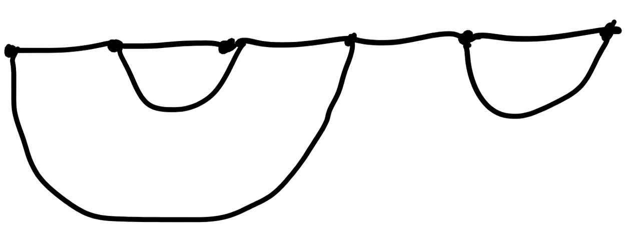
a =  , b =  ,

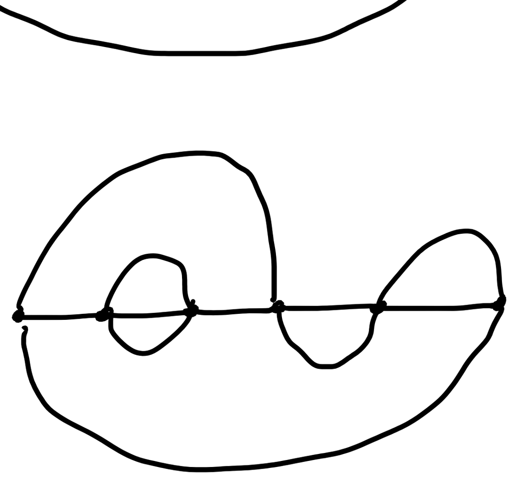
c =  , and so on.

**(Def)** $\bar{a}$ : For $a \in \mathcal{C}_n$, $\bar{a}$ denotes a reflection across the line containing the $2n$ pts:

b =  , $\bar{b}$ = 

⤳ We can stack matchings to produce circles:

e =  , b = 

⇒ $\bar{b}c$ : 

... let $\mathcal{F}(\bar{b}c) = Kh_2(\bar{b}a) = A^{\otimes k}$, $k$ = # of circles.

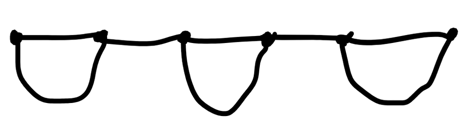
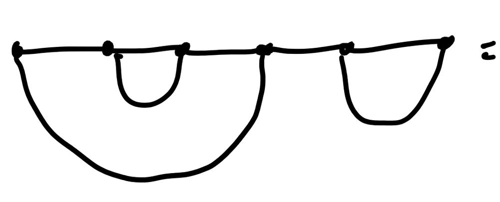
**(Def)** $a(H^n)b := q^n \mathcal{F}(\bar{b}a)$.

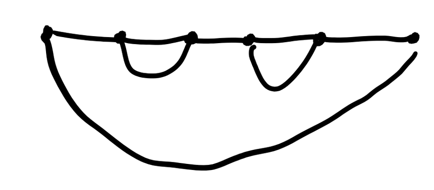
**(Def)** $H^n$ (Arc algebra): $H^n := \bigoplus_{a,b \in \mathcal{C}_n} a(H^n)b$

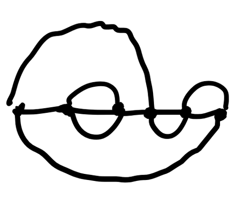
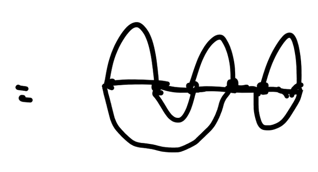
... the multiplication on $H^n$ is defined as follows:

$$a(H^n)b \otimes d(H^n)c \text{ "=" } \begin{cases} a(H^n)d & \text{if } b = d \\ 0 & \text{if } b \neq d \end{cases}$$

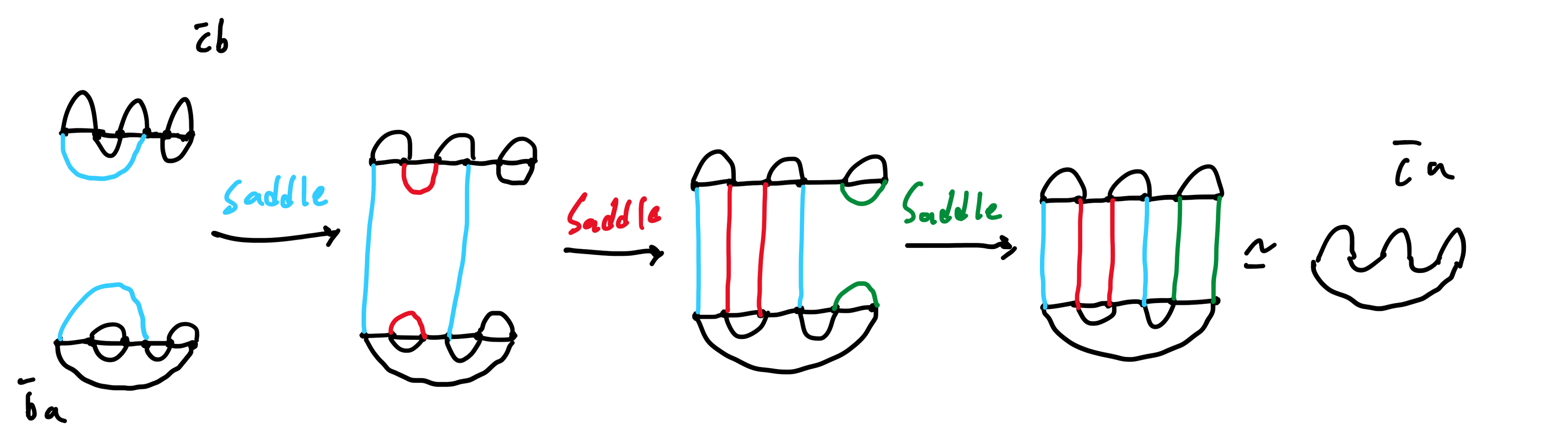
⤳ So multiplication is the zero map if $b \neq c$, but if $b = c$, then for $a, b, c \in \mathcal{C}_n$, there are saddle cobordisms $\bar{c}b \sqcup \bar{b}a \longrightarrow \bar{c}a$, multiplication is defined as the map induced by these.

**Ex.** Let c =  , b =  = d

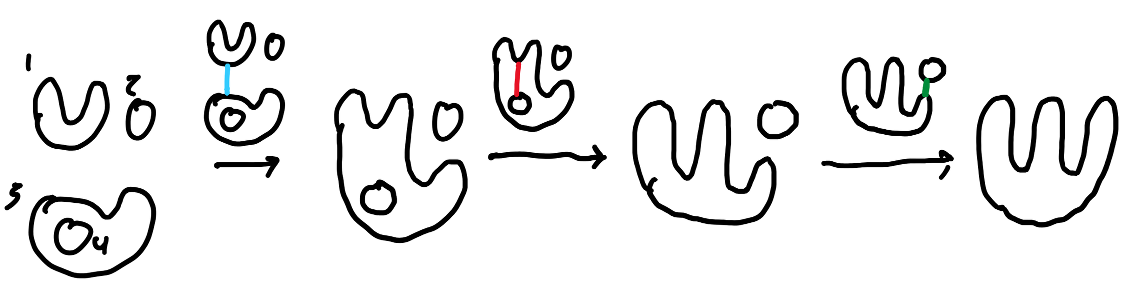
a = 

⇒ $\bar{b}a$ =  , $\bar{c}b$ = 

... since "b = d" our multiplication is non-trivial:




... So we have the following merges:



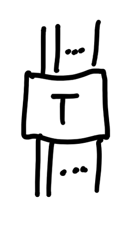
⇒ On $Kh_2$, we have:

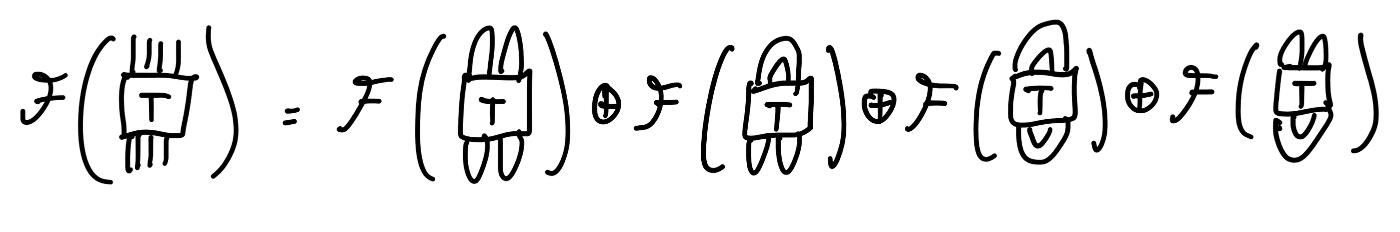
$$A_1 \otimes A_2 \otimes A_3 \otimes A_4 \xrightarrow{m_{1,3}} A_{13} \otimes A_2 \otimes A_4 \xrightarrow{m_{13,4}} A_{134} \otimes A_2 \xrightarrow{m_{2,134}} A_{2134}$$

$$A^{\otimes 2} \otimes A^{\otimes 2} \xrightarrow{m} A$$

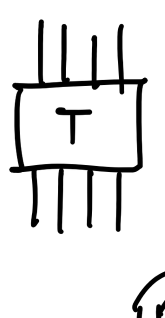
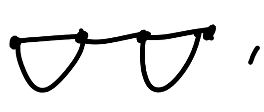
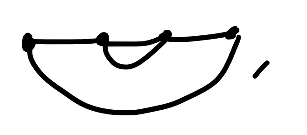
**(Def).** $a(\mathcal{F}(T))b$ : ... let $T$ be a $(2n, 2m)$-tangle, then for $a \in \mathcal{C}_m$, $b \in \mathcal{C}_n$, then we define $a(\mathcal{F}(T))b = CKh(\bar{b}Ta)$, the Khovanov complex associated to the knot/link $\bar{b}Ta$.

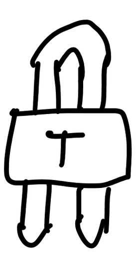
**(Def)** $\mathcal{F}(T) := \bigoplus_{\substack{a \in \mathcal{C}_m \\ b \in \mathcal{C}_n}} a(\mathcal{F}(T))b$

⤳ So for  , $\mathcal{F}(T)$ is the chain complex obtained by summing over all possible "crossingless matching" closures:

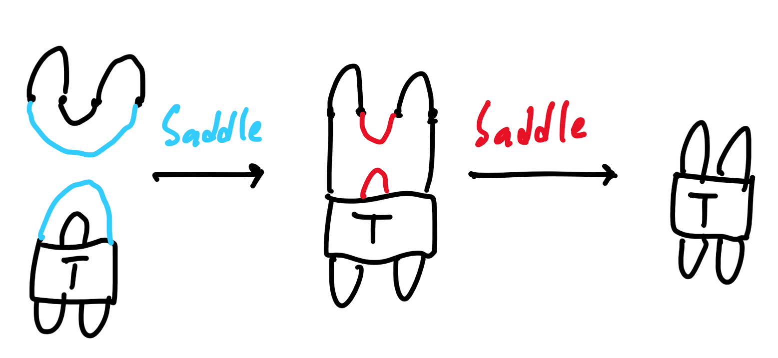


• **Note:** $\mathcal{F}(T)$ is an $(H^n, H^m)$-bimodule, with $H^n$ and $H^m$ acting on the top and bottom resp. The action can be understood visually on the crossingless matching level:

Given  , let a :  , b :  , then:

$\bar{b}Ta$ :  , for some composition of matchings $\bar{c}d$, if $d = b$, we once again have saddles:

(let $\bar{c} = \bar{a}$ for example...)

 = $\bar{a}Ta$

More to come..