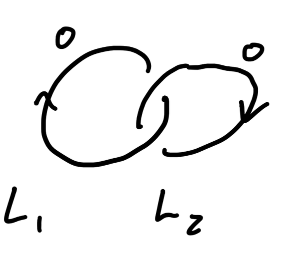
# Lasagna! - Cabled Kh

Tuesday, March 14, 2023 10:41 AM

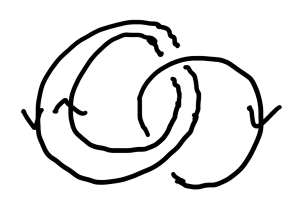
**Goal:** Define $\underline{Kh}_2$.

**(Def)** Cabled Link: Let $L = L_1 \cup \cdots \cup L_n \subset S^3$ be a framed link and $\kappa^{\pm} \in \mathbb{Z}^n$, then the $(\kappa^-, \kappa^+)$-cabled $L$ is the link obtained by $\kappa_i^-$ negatively oriented parallel strands & $\kappa_i^+$ positively oriented parallel strands to $L_i$.

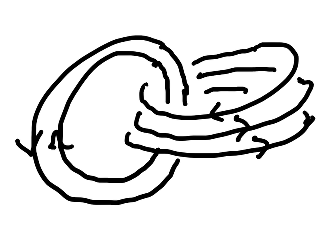
**Ex.**




$\rightsquigarrow ((1,0),(1,0))$ ($\kappa^-$, $\kappa^+$):



$\rightsquigarrow ((1,1),(1,2))$:



**Ex**


$\rightsquigarrow \overset{\kappa^-}{(1}, \overset{\kappa^+}{1)}$ : $T_{2,4}$ torus link (with all negative crossings)

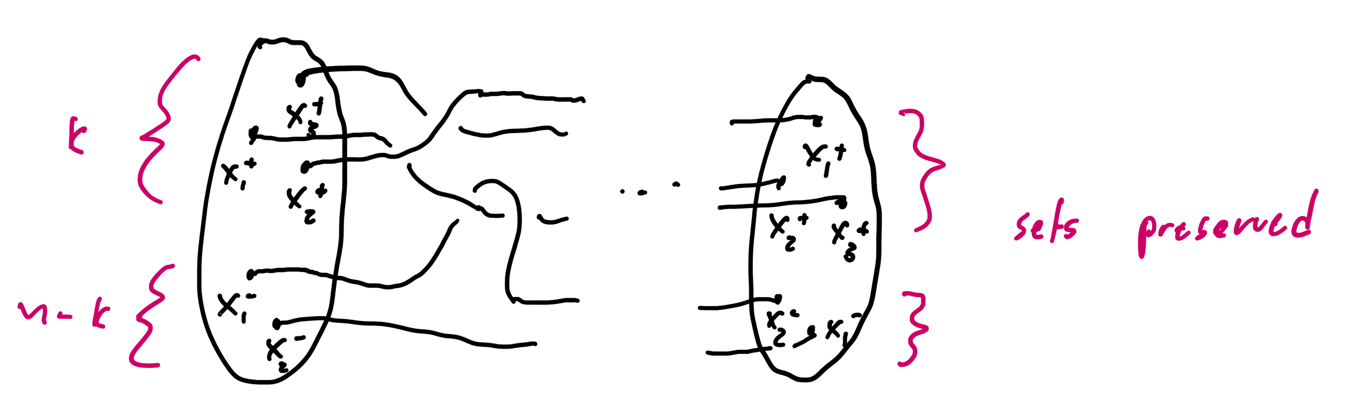
**• Note:** Modulo orientations, if $K$ is a knot, then $K(\kappa^-, \kappa^+)$ is the $(p(\kappa^- + \kappa^+), (\kappa^- + \kappa^+))$-cabled $K$, $p :=$ framing coefficient.

**• Note:** (Cables through marked $D^2$): Framing of $K$ is a longitudinal curve on a tubular nhood of $K$. Choose $x_1^-, \dots, x_{\kappa^-}^-, x_1^+, \dots, x_{\kappa^+}^+ \in D^2$, then consider $\varphi_p(S^1 \times \{x_1^-, \dots, x_{\kappa^+}^+\}) \subset N_K$, where $\varphi_p$ is the diffeo given by the framing, then:

$$K(\kappa^-, \kappa^+) = \varphi(S^1 \times \{x_1^-, \dots, x_{\kappa^+}^+\})$$

**(Def)** $B_{k,n-k}$: Let $B_n$ denote the braid group on $n$ strands, then $B_{k,n-k} \subset B_n$ is the subgroup that preserves, as a set, the first $k$ endpoints and last $n-k$ endpoints:

**Ex.** $b \in B_{2^-,3^+}$ as a cobordism:




(More precisely, $B_{k,n-k} = \mathcal{F}^{-1}(S_k \times S_{n-k})$ under the usual homomorphism $\mathcal{F} : B_n \to S_n$)

**(Def)** $\Sigma_b$ : ... for $b \in B_{\kappa^-,\kappa^+}$, let $\Sigma_b \subset S^1 \times D^2 \times [0,1]$ denote the cobordism obtained by $b \times S^1$, so $\Sigma_b : K(\kappa^-, \kappa^+) \to K(\kappa^-, \kappa^+)$.

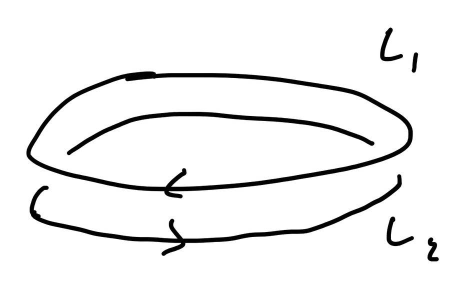
**(Def)** $\beta(b)$ : Let $\beta(b)$ denote the map on $Kh_2$ induced by $\Sigma_b$:

$$\beta(b) := Kh_2(\Sigma_b) : Kh_2(K(\kappa^-, \kappa^+)) \longrightarrow Kh_2(K(\kappa^-, \kappa^+))$$

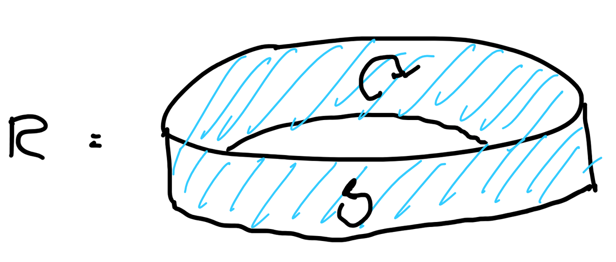
$\Rightarrow$ We have a group action $\beta : B_{\kappa^-,\kappa^+} \to Aut(Kh_2(K(\kappa^-, \kappa^+)))$.

$\rightsquigarrow$ For cabled Khovanov homology, there is a second cobordism to consider:

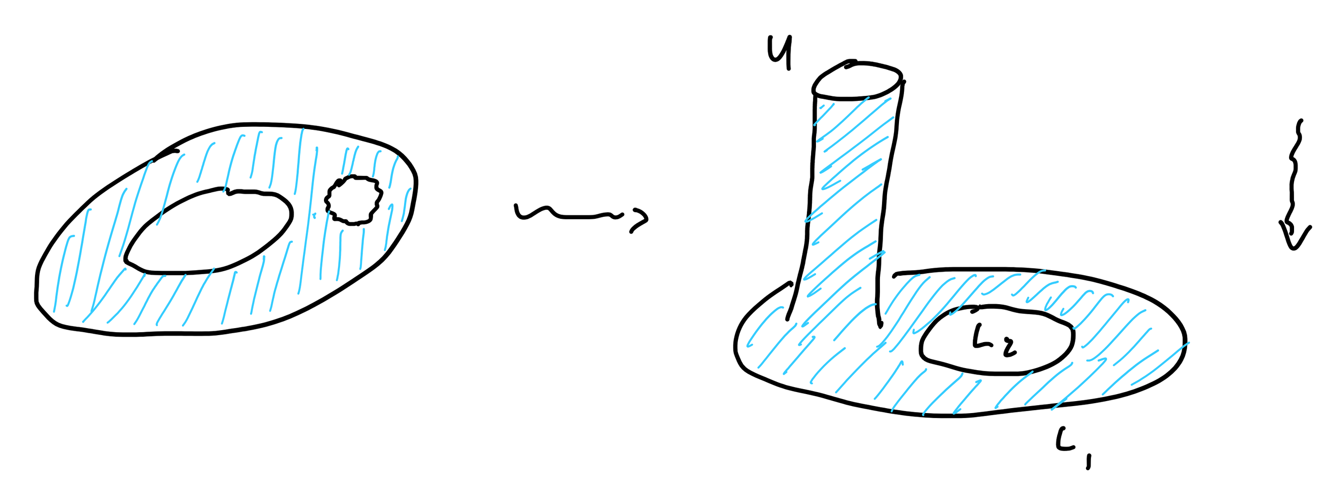
**(Def)** $Z$ : Given a cable of $K$, suppose there are two parallel strands w/ opposite orientation:




... Then they cobound a ribbon $R \subset S^3$:




... removing a disk yields a cobordism $U \to L_1 \sqcup L_2$, call it $Z$:




... $Z : K(\kappa^-, \kappa^+) \sqcup U \longrightarrow K(\kappa^- + 1, \kappa^+ + 1)$.

**• Note:** Once again we have a cobordism induced map:

$$Kh_2(Z) : Kh_2(K(\kappa^-, \kappa^+)) \otimes A \longrightarrow Kh_2(K(\kappa^- + 1, \kappa^+ + 1))$$

(Where $A := Kh_2(U) := \mathbb{Z}[X]/\langle X^2 \rangle$, $Kh_2(Z)$ shifts bi-grading by $(0,1)$)

**(Def)** $\psi^{[m]}$ := the "important part" of $Kh_2(Z)$:

$$\psi^{[m]} : Kh_2(K(\kappa^-, \kappa^+)) \longrightarrow Kh_2(K(\kappa^- + 1, \kappa^+ + 1))$$

... defined by: $\psi^{[0]}(v) = Kh_2(Z)(v \otimes 1)$
$\psi^{[1]}(v) = Kh_2(Z)(v \otimes X)$

$\rightsquigarrow \psi^{[m]}$ shifts bi-grading by $(0, 2m)$.

**(Def)** $Kh_{2,\alpha}(L)$ : Specify a homology class $\alpha \in H_2(W; \mathbb{Z}) \cong \mathbb{Z}^n$, let $\alpha_i^+ = \max\{\alpha_i, 0\}$, $\alpha_i^- = \min\{\alpha_i, 0\}$, and $|\alpha| = \sum \alpha_i$, then the cabled Khovanov homology at level $\alpha$ is:

$$\underline{Kh}_{2,\alpha}(L) := \left( \bigoplus_{r \in \mathbb{N}^n} q^{-2r-|\alpha|} Kh_2(L(r - \alpha^-, r + \alpha^+)) \right) \Big/ \sim$$

where "$\sim$": $\beta_i(b)(v) \sim v$, $\psi_i^{[0]} \sim 0$, $\psi_i^{[1]} \sim v$

**Ex.** For $L = U^0$, and $\alpha = 0$:

$$\underline{Kh}_{2,0}(L) = \left( Kh_2(\emptyset) \oplus q^{-2} Kh_2(\ldots) \oplus q^{-4} Kh_2(\ldots) \oplus \cdots \right) \Big/ \sim$$