Practice problems for the final exam

1. Prove the identity

$$1 \cdot 2 \cdot 3 + 2 \cdot 3 \cdot 4 + \ldots + n(n+1)(n+2) = \frac{1}{4}n(n+1)(n+2)(n+3).$$

2. Find the number of 100-element subsets of $\{1, \ldots, 2020\}$ containg at least one even integer.
3. Find the coefficient at $x^2y^2z^3$ in the expansion of $(x+y+z)^7$.
4. In a tournament there are 32 teams. In the first round they are grouped into 16 pairs. How many ways are there to group them into pairs?
5. Prove that $\binom{n}{k} - \binom{n}{k+1} + \binom{n}{k+2} = \binom{n-1}{k-1} + \binom{n-1}{k+2}$.
6. There are 20 identical coins. In how many ways one can distribute them between 6 people, such that each one gets at least one coin?
7. Recall that the *integer part* $\lfloor x \rfloor$ of a real number x is the maximal integer that is less than or equal to x. Given 500 real numbers in the interval $[0, 20]$, prove that there are at least 24 of them with the same integer part.
8. Are there two consecutive Fibonacci numbers F_n, F_{n+1} which are both divisible by 7? Find such numbers or prove that they do not exist.
9. A convex polyhedron P has v vertices, e edges and f faces. Consider the polyhedron P' obtained from P by truncating (cutting off) all vertices.
(a) Prove that P' has $2e$ vertices
(b) Prove that P' has $v + f$ faces
(c) Prove that P' has $3e$ edges.
10. Give an example of a planar graph where every vertex has degree 4.
11. Prove that there does not exist a planar graph where every vertex has degree 6.
12. A tree has 11 vertices.
(a) What is the maximal possible number of leaves for this tree?
(b) What is the minimal possible number of leaves for this tree?
13. A graph has 10 vertices and 40 edges.
(a) Prove that there is a vertex of degree at least 8.
(b) Prove that this graph is connected.
(c) Prove that this graph is not planar.
14. Consider the complete graph on (a) 5 (b) 6 (c) 7 vertices. Is there an Eulerian walk in this graph? Describe this walk or prove that it does not exist.