Practice problems for the final exam

1. Prove the identity
\[ 1 \cdot 2 \cdot 3 + 2 \cdot 3 \cdot 4 + \ldots + n(n+1)(n+2) = \frac{1}{4}n(n+1)(n+2)(n+3). \]

2. Find the number of 100-element subsets of \( \{1, \ldots, 2020\} \) containing at least one even integer.

3. Find the coefficient at \( x^2y^2z^3 \) in the expansion of \( (x + y + z)^7 \).

4. In a tournament there are 32 teams. In the first round they are grouped into 16 pairs. How many ways are there to group them into pairs?

5. Prove that \( \binom{n}{k} - \binom{n}{k+1} + \binom{n}{k+2} = \binom{n-1}{k-1} + \binom{n-1}{k+2} \).

6. There are 20 identical coins. In how many ways one can distribute them between 6 people, such that each one gets at least one coin?

7. Recall that the integer part \( \lfloor x \rfloor \) of a real number \( x \) is the maximal integer that is less than or equal to \( x \). Given 500 real numbers in the interval \([0, 20]\), prove that there are at least 24 of them with the same integer part.

8. Are there two consecutive Fibonacci numbers \( F_n, F_{n+1} \) which are both divisible by 7? Find such numbers or prove that they do not exist.

9. A convex polyhedron \( P \) has \( v \) vertices, \( e \) edges and \( f \) faces. Consider the polyhedron \( P' \) obtained from \( P \) by truncating (cutting off) all vertices.
   (a) Prove that \( P' \) has \( 2e \) vertices
   (b) Prove that \( P' \) has \( v + f \) faces
   (c) Prove that \( P' \) has \( 3e \) edges.

10. Give an example of a planar graph where every vertex has degree 4.

11. Prove that there does not exist a planar graph where every vertex has degree 6.

12. A tree has 11 vertices.
   (a) What is the maximal possible number of leaves for this tree?
   (b) What is the minimal possible number of leaves for this tree?

13. A graph has 10 vertices and 40 edges.
   (a) Prove that there is a vertex of degree at least 8.
   (b) Prove that this graph is connected.
   (c) Prove that this graph is not planar.

14. Consider the complete graph on (a) 5 (b) 6 (c) 7 vertices. Is there an Eulerian walk in this graph? Describe this walk or prove that it does not exist.