MAT 145, Spring 2020
Solutions to homework 1

1. (20 points) What is the number of positive integers with (a) at most n decimal digits (b) exactly n digits?

Solution:(a) (10 points) A number has at most n decimal digits if it is strictly less than 10^n. Therefore all such numbers are between 1 and $10^n - 1$, and there are $10^n - 1$ of them.

(b) (10 points) *First solution:* There are 9 options for the first digit, 10 options for the second digit, ... 10 options for the n-th digit, so there are $9 \cdot 10 \cdot \cdots \cdot 10 = 9 \cdot 10^{n-1}$ such numbers.

Second solution: Let K_n be the number of positive integers with exactly n digits. Clearly, $K_1 = 9$. If $a_1 \ldots a_n$ is an integer with exactly n digits and $n > 1$, then $a_1 \ldots a_{n-1}$ is an integer with exactly $(n-1)$ digits, and a_n could be any of 10 digits. We get a recursion

$$K_1 = 9,\ K_n = 10K_{n-1}\ (n > 1).$$

It can be easily solved and gives $K_n = 9 \cdot 10^{n-1}$.

2. (20 points) Prove by induction that $1 + 2 + \ldots + n = \frac{n(n+1)}{2}$.

Solution: Base: For $n = 1$ we get $1 = \frac{1 \cdot 2}{2}$.

Step: Assume that $1 + 2 + \ldots + n = \frac{n(n+1)}{2}$. Then

$$1 + 2 + \ldots + n + (n+1) = \frac{n(n+1)}{2} + (n+1) = \frac{n(n+1)}{2} + \frac{2(n+1)}{2} =$$
$$\frac{n(n+1) + 2(n+1)}{2} = \frac{(n+1)(n+2)}{2}.$$

The desired equation holds for $n = 1$ (Base), and if it holds for n then it holds for $(n+1)$ (Step), so by induction we conclude that it holds for all n.

3. (20 points) Prove by induction that

$$1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \ldots + (n-1) \cdot n = \frac{(n-1)n(n+1)}{3}.$$

Solution: Base: For $n = 2$ we get $1 \cdot 2 = \frac{1 \cdot 2 \cdot 3}{3}$ (note that the left hand side is not defined for $n = 1$).

Step: Assume that $1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \ldots + (n-1) \cdot n = \frac{(n-1)n(n+1)}{3}$.. Then

$$1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \ldots + (n-1) \cdot n + n \cdot (n+1) = \frac{(n-1)n(n+1)}{3} + n(n+1) =$$
$$\frac{(n-1)n(n+1)}{3} + \frac{3n(n+1)}{3} = \frac{(n-1)n(n+1) + 3n(n+1)}{3} =$$
$$\frac{(n-1+3)n(n+1)}{3} = \frac{n(n+1)(n+2)}{3}.$$

The desired equation holds for $n = 1$ (Base), and if it holds for n then it holds for $(n+1)$ (Step), so by induction we conclude that it holds for all n.

4. (20 points) Prove by induction that

$$1 + 2 + \ldots + 2^n = 2^{n+1} - 1.$$

Base: For $n = 0$ we get $1 = 2^1 - 1$.

Step: Assume that $1 + 2 + \ldots + 2^n = 2^{n+1} - 1$.. Then

$$1 + 2 + \ldots + 2^n + 2^{n+1} = (2^{n+1} - 1) + 2^{n+1} = 2 \cdot 2^{n+1} - 1 = 2^{n+2} - 1.$$

The desired equation holds for $n = 1$ (Base), and if it holds for n then it holds for $(n+1)$ (Step), so by induction we conclude that it holds for all n.

5. (20 points) A drawer contains 6 pairs of black, 5 pairs of white, 5 pairs of red and 4 pairs of green socks.
(a) How many single socks do we have to take out to make sure that we take two socks with the same color?
(b) How many single socks do we have to take out to make sure that we take two socks with different colors?

Solution: (a) (10 points) We need to take 5 socks: indeed, we have 4 different colors, so by "pigeonhole principle" there should be two socks out of 5 with the same color.

(b) (10 points) Let us prove that if we take out 13 socks there will be two socks with different colors. Assume the contrary, and all of them are of the same color. If they are all black, there are at most 12 of them. If they are all white, there are at most 10 of them. If they are all red, there are at most 10 of them. If they are all green, there are at most 8 of them. Contradiction, so there are two socks with different colors.