MAT 145, Spring 2020
Solutions to homework 2

1. (20 points) How many numbers between 1 and 2020 are divisible by 3, 7 or 11?

Solution: Let $\lfloor x \rfloor$ denote the integer part of x, that is, maximal integer less than or equal to x. We have $673 = \lfloor 2020/3 \rfloor$ numbers between 1 and 2020 divisible by 3, $288 = \lfloor 2020/7 \rfloor$ divisible by 7, $183 = \lfloor 2020/11 \rfloor$ divisible by 11. Furthermore, a number is divisible both by 3 and 7 if and only if it is divisible by 21, and there are $96 = \lfloor 2020/21 \rfloor$ such numbers. A number is divisible by 3 and 11 if it is divisible by 33, and there are $61 = \lfloor 2020/33 \rfloor$ such numbers. A number is divisible by 7 and 11 if it is divisible by 77, and there are $26 = \lfloor 2020/77 \rfloor$ such numbers. Finally, a number is divisible by 3,7 and 11 if it is divisible by $3 \cdot 7 \cdot 11 = 231$, and there are just $8 = \lfloor 2020/231 \rfloor$ such numbers.

Now by the inclusion-exclusion formula we get the total number if integers divisible by 3, 7 or 11:

$$673 + 288 + 183 - 96 - 61 - 26 + 8 = 969.$$

Answer: 969.

2. (20 points) Prove that

$$\binom{n}{k} = \binom{n-2}{k} + 2\binom{n-2}{k-1} + \binom{n-2}{k-2}.$$

Solution: By applying the recursive formula twice we get:

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1} =$$

$$\left[\binom{n-2}{k} + \binom{n-2}{k-1}\right] + \left[\binom{n-2}{k-1} + \binom{n-2}{k-2}\right] =$$

$$\binom{n-2}{k} + 2\binom{n-2}{k-1} + \binom{n-2}{k-2}.$$

3. (20 points) Find the value of the following sum:

$$0 \cdot \binom{n}{0} + 1 \cdot \binom{n}{1} + 2 \cdot \binom{n}{2} + \ldots + n \cdot \binom{n}{n}$$

for all n. Compute the values for small n first, guess the general formula and then prove it for all n.

Solution 1: Observe that

$$k\binom{n}{k} = k\frac{n!}{k!(n-k)!} = k \cdot \frac{n(n-1)!}{k(k-1)!(n-k)!} =$$

$$n \cdot \frac{(n-1)!}{(k-1)!(n-k)!} = n\binom{n-1}{k-1}.$$

Therefore

$$0 \cdot \binom{n}{0} + 1 \cdot \binom{n}{1} + 2 \cdot \binom{n}{2} + \ldots + n \cdot \binom{n}{n} =$$

$$0 + n \cdot \binom{n-1}{0} + n \cdot \binom{n-1}{1} + \cdots + n \cdot \binom{n-1}{n-1} =$$

$$n\left[\binom{n-1}{0} + \binom{n-1}{1} + \ldots + \binom{n-1}{n-1}\right] = n \cdot 2^{n-1}.$$

Solution 2: Let us prove by induction in n that

$$0 \cdot \binom{n}{0} + 1 \cdot \binom{n}{1} + 2 \cdot \binom{n}{2} + \ldots + n \cdot \binom{n}{n} = n \cdot 2^{n-1}.$$

Base: For $n = 1$ we get $0\binom{1}{0} + 1\binom{1}{1} = 1 = 1 \cdot 2^0$.

Step: Suppose that

$$0 \cdot \binom{n}{0} + 1 \cdot \binom{n}{1} + 2 \cdot \binom{n}{2} + \ldots + n \cdot \binom{n}{n} = n \cdot 2^{n-1}.$$

then

$$0\cdot\binom{n+1}{0}+1\cdot\binom{n+1}{1}+2\cdot\binom{n+1}{2}+\ldots+n\cdot\binom{n+1}{n}+(n+1)\binom{n+1}{n+1} =$$

$$1\left[\binom{n}{0} + \binom{n}{1}\right] + 2\left[\binom{n}{1} + \binom{n}{2}\right] + 3\left[\binom{n}{2} + \binom{n}{3}\right] + \ldots$$

$$+n\left[\binom{n}{n-1} + \binom{n}{n}\right] + (n+1)\binom{n}{n} =$$

$$1\binom{n}{0} + 3\binom{n}{1} + 5\binom{n}{2} + \ldots + (2n+1)\binom{n}{n} =$$

$$2\left[0 \cdot \binom{n}{0} + 1 \cdot \binom{n}{1} + 2 \cdot \binom{n}{2} + \ldots + n \cdot \binom{n}{n}\right] +$$

$$\left[\binom{n}{0} + \binom{n}{1} + 2 \cdot \binom{n}{2} + \ldots + \cdot\binom{n}{n}\right] =$$

$$2 \cdot n \cdot 2^{n-1} + 2^n = n \cdot 2^n + 2^n = (n+1)2^n.$$

4. (20 points) The city has a rectangular grid of streets. 6 avenues go from north to south, and 30 streets go from east to west. How many ways are there to get from the corner of 1st street and 1st avenue to the corner of 30th street and 6th avenue?

Solution: Any such path is a sequance of 29 North steps and 5 East steps, and we can encode it in a sequence of 29N's and 5E's. The number of such sequences equals

$$\binom{29+5}{5} = \binom{34}{5}.$$

5. (20 points) Prove the identity for all n:

$$1 + 2\binom{n}{1} + 4\binom{n}{2} + \ldots + 2^n\binom{n}{n} = 3^n.$$

Solution: By Binomial Theorem we have

$$3^n = (1+2)^n = 1^n + 1^{n-1}2\binom{n}{1} + 1^{n-2}2^2\binom{n}{2} + \ldots + 2^n\binom{n}{n} =$$

$$1 + 2\binom{n}{1} + 4\binom{n}{2} + \ldots + 2^n\binom{n}{n}.$$