MAT 145, Spring 2020
Solutions to Homework 3

1. (20 points) How many anagrams can you make from the word COMBINATORICS ?

Solution: We have 13 letters, among them 2 C's, 2 O's, 1 M, 1 B, 2 I's, 1 N, 1 A, 1 T, 1 R and 1 S. By multinomial formula we get

$$\text{The number of anagrams} = \frac{13!}{2!2!1!1!2!1!1!1!1!1!} = \frac{13!}{8}.$$

2. (20 points) In how many ways can you distribute n pennies to r children if each child is supposed to get at least 2?

Solution: We first give each child two pennies, and need to distribute the remaining $n-2r$ between r children. We use "stars and bars" method: there are $(n-2r)$ stars and $(r-1)$ bars which distribute them in r groups. The total number of positions for stars and bars is $(n-2r)+(r-1) = n-r-1$, and we need to choose $r-1$ positions for bars, so the final answer is

$$\binom{n-r-1}{r-1}.$$

3. (20 points) Find the value of k for which $k\binom{99}{k}$ is the largest.

Solution 1: We have $k\binom{99}{k} = k\frac{99!}{k!(99-k)!} = 99\frac{98!}{(k-1)!(99-k)!} = 99\binom{98}{k-1}$. It is the largest when $\binom{98}{k-1}$ is the largest which happens when $k-1 = 98/2 = 49$, so $k = 50$.

Solution 2: Let us compare the values of $k\binom{99}{k}$ and $(k-1)\binom{99}{k-1}$:

$$\frac{k\binom{99}{k}}{(k-1)\binom{99}{k-1}} = k \cdot \frac{99!}{k!(99-k)!} \cdot \frac{1}{k-1} \cdot \frac{(k-1)! \cdot (100-k)!}{99!} =$$

$$\frac{k \cdot 99! \cdot (k-1)! \cdot (99-k)!(100-k)}{k \cdot (k-1)! \cdot (99-k)! \cdot (k-1)99!} = \frac{100-k}{k-1}.$$

We have

$$\frac{100-k}{k-1} > 1 \Leftrightarrow 100-k > k-1 \Leftrightarrow 101 > 2k \Leftrightarrow k \le 50.$$

This means that the values $k\binom{99}{k}$ increase for $k \le 50$ and decrease for $k > 50$, so the largest value is at $k = 50$.

4. (20 points) Find the coefficient at x^2yzw^7 in the expansion of

$$(x+y+z+w)^{11}.$$

Solution: By Multinomial Theorem the coefficient equals

$$\frac{11!}{2!1!1!7!} = \frac{11 \cdot 10 \cdot 9 \cdot 8}{2}.$$

5. (20 points) Find the sum

$$\binom{3}{3} + \binom{4}{3} + \ldots + \binom{n}{3}$$

for all $n \geq 3$.

Solution: We prove by induction that

$$\binom{3}{3} + \binom{4}{3} + \ldots + \binom{n}{3} = \binom{n+1}{4}.$$

Base: For $n = 3$ we get $\binom{3}{3} = 1 = \binom{4}{4}$.

Step: Suppose that $\binom{3}{3} + \binom{4}{3} + \ldots + \binom{n}{3} = \binom{n+1}{4}$. Then

$$\binom{3}{3} + \binom{4}{3} + \ldots + \binom{n}{3} + \binom{n+1}{3} = \binom{n+1}{4} + \binom{n+1}{3} = \binom{n+2}{4}.$$