MAT 145, Spring 2020
Solutions to homework 4

1. (20 points) Prove the identity for all $n \geq k \geq m$:

$$\binom{n}{k}\binom{k}{m} = \binom{n}{m}\binom{n-m}{k-m}$$

Solution: We have

$$\binom{n}{k}\binom{k}{m} = \frac{n!}{k!(n-k)!} \cdot \frac{k!}{m!(k-m)!} = \frac{n!}{(n-k)!m!(k-m)!},$$

$$\binom{n}{m}\binom{n-m}{k-m} = \frac{n!}{m!(n-m)!} \cdot \frac{(n-m)!}{(k-m)!(n-k)!} = \frac{n!}{m!(k-m)!(n-k)!}.$$

2. (20 points) Use the result of Problem 1 above to prove the identity for all $n \geq m$:

$$\sum_{k=m}^{n} \binom{n}{k}\binom{k}{m} = \binom{n}{m}\binom{m}{m} + \ldots + \binom{n}{n}\binom{n}{m} = \binom{n}{m} 2^{n-m}.$$

Solution: Using Problem 1 we can rewrite

$$\sum_{k=m}^{n} \binom{n}{k}\binom{k}{m} = \sum_{k=m}^{n} \binom{n}{m}\binom{n-m}{k-m} =$$

$$\binom{n}{m}\binom{n-m}{0} + \binom{n}{m}\binom{n-m}{1} + \ldots + \binom{n}{m}\binom{n-m}{n-m} =$$

$$\binom{n}{m}\left(\binom{n-m}{0} + \binom{n-m}{1} + \ldots + \binom{n-m}{n-m}\right) = \binom{n}{m} \cdot 2^{n-m}.$$

3. (20 points) Prove that the Fibonacci number F_{3n} is even for all n.

Solution 1: Let us prove by induction that F_{3n} is even, F_{3n+1} is odd and F_{3n+2} is odd.

Base: We have $F_0 = 0, F_1 = F_2 = 1$.

Step: Suppose that F_{3n} is even, F_{3n+1} is odd and F_{3n+2} is odd. Then $F_{3n+3} = F_{3n+1} + F_{3n+2}$ is a sum of two odd numbers, so it is even. Similarly, $F_{3n+4} = F_{3n+3} + F_{3n+2}$ is a sum of even and odd number, so it is odd, and $F_{3n+5} = F_{3n+3} + F_{3n+4}$ is a sum of even and odd number, so it is odd.

Solution 2: Let us prove by induction that F_{3n} is even.

Base: $F_3 = 2$ is even

Step: Assume that F_{3n} is even. Then

$$F_{3n+3} = F_{3n+2} + F_{3n+1} = (F_{3n} + F_{3n+1}) + F_{3n+1} = F_{3n} + 2F_{3n+1}.$$

Since F_{3n} is even, and $2F_{3n+1}$ is even, F_{3n+3} is even.

4. (20 points) Prove the identity for Fibonacci numbers:

$$F_1 + F_3 + \ldots + F_{2n-1} = F_{2n}$$

Solution: We prove it by induction in n.
Base: For $n = 1$ we get $F_1 = F_2 = 1$
Step: Assume that

$$F_1 + F_3 + \ldots + F_{2n-1} = F_{2n}$$

Then

$$F_1 + F_3 + \ldots + F_{2n-1} + F_{2n+1} = F_{2n} + F_{2n+1} = F_{2n+2}.$$

5. (20 points) In how many ways can you cover a $2 \times n$ chessboard with dominoes (2×1 rectangles)?

Solution 1: Let A_n be the number of ways to tile the board with dominoes. We prove by induction in n that $A_n =_{n+1}$.
Base: $A_1 = 1 = F_2$, $A_2 = 2 = F_3$.
Step: Suppose that we need to tile $2 \times (n+1)$ board. Then either there is a vertical domino on the right end, and there are A_n ways to tile the remaining $2 \times n$ rectangle, or there is a pair of horizontal dominoes on the right, and there are A_{n-1} ways to tile the remaining $2 \times (n-1)$ rectangle. Therefore

$$A_{n+1} = A_n + A_{n-1} = F_{n+1} + F_n = F_{n+2}.$$

Solution 2: Given a tiling, let us group its columns as following. A vertical domino is an "ordinary" column while a pair of horizontal dominoes is a "double column". If there are k "double" columns then there are $n - 2k$ "ordinary" ones and $n - k$ columns in total. We need to choose k "double "columns out of these $n - k$, so the number of tilings with k "double" columns equals $\binom{n-k}{k}$. The total number of tilings equals $\sum_k \binom{n-k}{k} = F_{n+1}$, as we proved in class.