# MAT 145, Spring 2020
# Solutions to homework 5

Problems 1-3 are about *Lucas numbers* defined by

$$L_1 = 1, L_2 = 3 \text{ and } L_{n+1} = L_n + L_{n-1}$$

**1.** (20 points) Prove that $L_n = 2F_{n+1} - F_n$.

**Solution:** We prove it by induction in $n$.

Base: For $n = 1$ we get $2F_2 - F_1 = 2 \cdot 1 - 1 = 1 = L_1$ and for $n = 2$ we get $2F_3 - F_2 = 2 \cdot 2 - 1 = 3 = L_2$.

Step: Suppose that $L_n = 2F_{n+1} - F_n$ and $L_{n-1} = 2F_n - F_{n-1}$. Then

$$L_{n+1} = L_n + L_{n-1} = 2F_{n+1} - F_n + 2F_n - F_{n-1} =$$

$$2(F_{n+1} + F_n) - (F_n + F_{n-1}) = 2F_{n+2} - F_{n+1}.$$

**2.** (20 points) Prove that

$$L_n = \left(\frac{1+\sqrt{5}}{2}\right)^n + \left(\frac{1-\sqrt{5}}{2}\right)^n$$

**Solution 1:** We use the result of previous problem and the formula for Fibonacci numbers

$$F_n = \frac{1}{\sqrt{5}}(\varphi^n - \overline{\varphi}^n),\ \varphi = \frac{1+\sqrt{5}}{2},\ \overline{\varphi} = \frac{1-\sqrt{5}}{2}.$$

Then by Problem 1 we have

$$L_n = 2F_{n+1} - F_n = \frac{1}{\sqrt{5}}(2\varphi^{n+1} - 2\overline{\varphi}^n - \varphi^n + \overline{\varphi}^n) =$$

$$\frac{1}{\sqrt{5}}((2\varphi - 1)\varphi^n - (2\overline{\varphi} - 1)\overline{\varphi}^n) = \frac{1}{\sqrt{5}}(\sqrt{5}\varphi^n + \sqrt{5}\overline{\varphi}^n) = \varphi^n + \overline{\varphi}^n.$$

Here we used that $2\varphi - 1 = \sqrt{5}$ and $2\overline{\varphi} - 1 = -\sqrt{5}$.

**Solution 2:** We prove it by induction in $n$.

Base: For $n = 1$ we get $\varphi + \overline{\varphi} = 1 = L_1$, and for $n = 2$ we get

$$\varphi^2 + \overline{\varphi}^2 = \frac{1 + 2\sqrt{5} + 5}{4} + \frac{1 - 2\sqrt{5} + 5}{4} = \frac{1 + 5 + 1 + 5}{4} = 3 = L_2.$$

Step: Suppose that $L_n = \varphi^n + \overline{\varphi}^n$ and $L_{n-1} = \varphi^{n-1} + \overline{\varphi}^{n-1}$. Then

$$L_{n+1} = L_n + L_{n-1} = \varphi^n + \overline{\varphi}^n + \varphi^{n-1} + \overline{\varphi}^{n-1} =$$

$$(\varphi + 1)\varphi^{n-1} + (\overline{\varphi} + 1)\overline{\varphi}^{n-1} = \varphi^{n+1} + \overline{\varphi}^{n+1}.$$

Here we used that $\varphi + 1 = \varphi^2$ and $\overline{\varphi} + 1 = \overline{\varphi}^2$.

**3.** (20 points) A round flower bed has $n \geq 2$ slots numbered from 1 to $n$. Prove that there are $L_n$ ways to plant tulips and lilies in such a way that there are no two lilies planted next to each other (but two tulips can be planted next to each other).

**Solution:** Let us label the slots from 1 to $n$ such that 1 is next to $n$. We abbreviate tulips by T and lilies by L.

If we place T at slot 1 then we can fill the slots $2, \dots, n$ by an arbitrary sequence of L's and T's without LL. The number of such sequences equals $F_{(n-1)+2} = F_{n+1}$.

If we place L at slot 1 then both slots 2 and $n$ should contain T, and we can fill the slots $3, \dots, n-1$ by an arbitrary sequence of L's and T's without LL. The number of such sequences equals $F_{(n-3)+2} = F_{n-1}$.

In total, we get

$$F_{n+1} + F_{n-1} = F_{n+1} + F_{n+1} - F_n = 2F_{n+1} - F_n = L_n$$

options.

**4.** (20 points) Prove the identity $F_{a+b+1} = F_{a+1}F_{b+1} + F_aF_b$.

**Solution 1:** We fix $a$ and prove it by induction in $b$.

Base: For $b = 0$ we get $F_{a+1}F_1 + F_aF_0 = F_{a+1} \cdot 1 + F_a \cdot 0 = F_{a+1}$. For $b = 1$ we get $F_{a+1}F_2 + F_aF_1 = F_{a+1} \cdot 1 + F_a \cdot 1 = F_{a+2}$.

Step: Assume that the identity holds for $b$ and $(b-1$, that is,

$$F_{a+b+1} = F_{a+1}F_{b+1} + F_aF_b, \;\; F_{a+b} = F_{a+1}F_b + F_aF_{b-1}$$

Then

$$F_{a+b+2} = F_{a+b+1} + F_{a+b} = F_{a+1}F_{b+1} + F_aF_b + F_{a+1}F_b + F_aF_{b-1} =$$
$$F_{a+1}(F_{b+1} + F_b) + F_a(F_b + F_{b-1}) = F_{a+1}F_{b+2} + F_aF_{b+1}.$$

**Solution 2:** We know that $F_{a+b+1}$ is the number of binary sequences of length $a + b + 1 - 2 = (a-1) + 1 + (b-1)$ without 11. Given such a sequence, we can look at its $a$th position.

If we have 0 at $a$-th position, then to the left and right of it there are two sequences of length $(a-1)$ and $(b-1)$ without 11, and the number of such sequences equals $F_{(a-1)+2}F_{(b-1)+2} = F_{a+1}F_{b+1}$.

If we have 1 at $a$th position, then it must be surrounded by 0 at either side, and we are left with two sequences of length $(a-2)$ and $(b-2)$ without 11, the number of such sequences equals $F_{(a-2)+2}F_{(b-2)+2} = F_aF_b$.

In total, we get $F_{a+b+1} = F_{a+1}F_{b+1} + F_aF_b$.

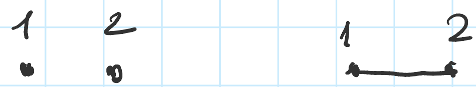
**5.** (20 points) Draw all graphs with 2, 3 and 4 vertices.

Lecture 16 (5/6)

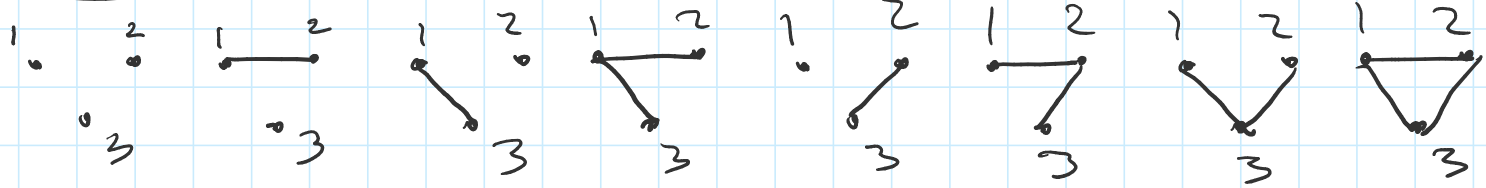
Wednesday, May 6, 2020 11:45 AM

# Graphs on 2,3,4 vertices:

**n=2**



**n=3**



**n=4**