MAT 145, Spring 2020
Solutions to homework 6

1. (20 points) A graph has two connected components with $m$ and
   $n - m$ vertices. What is the maximal possible number of edges in this
   graph?

   Solution: The maximal number of edges in each component equals
   $\binom{m}{2}$ and $\binom{n - m}{2}$, so the maximal total number is
   $\binom{m}{2} + \binom{n - m}{2}$.

2. (20 points) Use problem 1 to prove that a graph with $n$ vertices and
   more than $\binom{n - 1}{2}$ edges is connected.

   Solution: Let us first prove that
   $\binom{m}{2} + \binom{n - m}{2} \leq \binom{n - 1}{2}$ for $1 \leq m \leq n - 1$. Indeed, let
   
   $f(m) = \binom{m}{2} + \binom{n - m}{2} = \frac{m(m - 1)}{2} + \frac{(n - m)(n - m - 1)}{2} = 
   \frac{m^2 - mn}{2} + \frac{n^2 - mn - n + m^2 + m}{2} = 
   \frac{2m^2 - 2mn}{2} + \frac{n^2 - n}{2} = m(m - n) + \binom{n}{2}$.

   Then $f'(m) = 2m - n$, so $f(m)$ decreases for $m < n/2$ and increases
   for $m > n/2$. Therefore the maximal value of $f(m)$ on $[1, n - 1]$ equals
   $f(1) = f(n - 1) = \binom{n - 1}{2}$.

   Now let us prove the problem by contradiction. Assume that a graph
   $G$ has more than $\binom{n - 1}{2}$ edges is disconnected. Then we can break it
   into two components with $m$ and $n - m$ vertices respectively (for some
   $m$), and the total number of edges is at most $\binom{m}{2} + \binom{n - m}{2} \leq \binom{n - 1}{2}$. Contradiction, hence $G$ is connected.

3. (20 points) Are there graphs on 6 vertices with degrees (a) 0, 1, 2, 3, 4, 5?
   (b) 2, 3, 3, 3, 3, 3? Either draw the graphs or prove that they do not exist.

   Solution: We know that the number of vertices of odd degree is
   even (since the total sum of degrees equals $2 \cdot$ (number of edges). But
   in (a) there are 3 vertices of odd degree and in (b) there are 5 vertices
   of odd degree, so such graphs do not exist.

4. (20 points) Prove that in any graph (with at least 2 vertices) there
   are two vertices of the same degree.

   Solution: We use pigeonhole principle here. Assume that the graph
   has $n$ vertices and all vertices have different degrees. Since the maximal
   possible degree is $(n - 1)$, there are vertices of all possible degrees from
0 to $(n - 1)$. In particular, there is a vertex $x$ of degree 0, which is not connected to any other vertex, and there is a vertex $y$ of degree $n - 1$, which is connected to every other vertex. But this means that $x$ is connected to $y$, contradiction.

5. (20 points) Which of the following graphs have Eulerian walks? Find an Eulerian walk or prove that it does not exist.

Solution: By Euler’s theorem, a graph has an Eulerian walk if and only if it has at most 2 vertices of odd degree. In the first graph there are 4 vertices of odd degree, so there is no Eulerian walk. The other two graphs have 0 and 2 vertices of odd degree, so they have Eulerian walks. The walks are shown below.

![Graphs](image-url)