MAT 145, Spring 2020
Solutions to homework 6

1. (20 points) A graph has two connected components with m and $n-m$ vertices. What is the maximal possible number of edges in this graph?

Solution: The maximal number of edges in each component equals $\binom{m}{2}$ and $\binom{n-m}{2}$, so the maximal total number is $\binom{m}{2}+\binom{n-m}{2}$.

2. (20 points) Use problem 1 to prove that a graph with n vertices and more than $\binom{n-1}{2}$ edges is connected.

Solution: Let us first prove that $\binom{m}{2}+\binom{n-m}{2}\le\binom{n-1}{2}$ for $1\le m\le n-1$. Indeed, let

$$f(m)=\binom{m}{2}+\binom{n-m}{2}=\frac{m(m-1)}{2}+\frac{(n-m)(n-m-1)}{2}=$$

$$\frac{m^2-m}{2}+\frac{n^2-mn-n-mn+m^2+m}{2}=$$

$$\frac{2m^2-2mn}{2}+\frac{n^2-n}{2}=m(m-n)+\binom{n}{2}.$$

Then $f'(m)=2m-n$, so $f(m)$ decreases for $m<n/2$ and increases for $m>n/2$. Therefore the maximal value of $f(m)$ on $[1,n-1]$ equals $f(1)=f(n-1)=\binom{n-1}{2}$.

Now let us prove the problem by contradiction. Assume that a graph G has more than $\binom{n-1}{2}$ edges is disconnected. Then we can break it into two components with m and $n-m$ vertices respectively (for some m), and the total number of edges is at most $\binom{m}{2}+\binom{n-m}{2}\le\binom{n-1}{2}$. Contradiction, hence G is connected.

3. (20 points) Are there graphs on 6 vertices with degrees (a) $0,1,2,3,4,5$? (b) $2,3,3,3,3,3$? Either draw the graphs or prove that they do not exist.

Solution: We know that the number of vertices of odd degree is even (since the total sum of degrees equals $2\cdot$(number of edges). But in (a) there are 3 vertices of odd degree and in (b) there are 5 vertices of odd degree, so such graphs do not exist.

4. (20 points) Prove that in any graph (with at least 2 vertices) there are two vertices of the same degree.

Solution: We use pigeonhole principle here. Assume that the graph has n vertices and all vertices have different degrees. Since the maximal possible degree is $(n-1)$, there are vertices of all possible degrees from

0 to $(n-1)$. In particular, there is a vertex x of degree 0, which is not connected to any other vertex, and there is a vertex y of degree $n-1$, which is connected to every other vertex. But this means that x is connected to y, contradiction.

5. (20 points) Which of the following graphs have Eulerian walks? Find an Eulerian walk or prove that it does not exist.

Solution: By Euler's theorem, a graph has an Eulerian walk if and only if it has at most 2 vertices of odd degree. In the first graph there are 4 vertices of odd degree, so there is no Eulerian walk. The other two graphs have 0 and 2 vertices of odd degree, so they have Eulerian walks. The walks are shown below.