MAT 145, Spring 2020
Solutions to homework 7

1. (20 points) Prove that a graph with n nodes and m edges ($m < n$) has at least $n - m$ connected components.

Solution 1: We prove it by induction in m. If $m = 0$ then we have n vertices and no edges, so n connected components. Suppose that we proved this for ane graph with m edges, consider a graph G with $(m + 1)$ edges. Let G' be a subgraph of G with one edge e removed. By the assumption of induction, G' has k connected components and $k \geq n - m$. The edge e either connects two components of G' and then G has $k - 1$ components, or it connects two vertices in the same connected components of G' and then G has k components. In any case, G has at least $k - 1 \geq n - m - 1 = n - (m + 1)$ components.

Solution 2: If G is a connected graph then it has at least $n - 1$ edges. Indeed, a spanning tree in G has exactly $n - 1$ edges.

Now suppose that G has k connected components with $n_1, \ldots, n_k$ vertices and $m_1, \ldots, m_k$ edges. Then by the above $m_i \geq n_i - 1$, so

$$m = m_1 + \ldots + m_k \geq (n_1 - 1) + \ldots + (n_k - 1) = (n_1 + \ldots + n_k) - k = n - k.$$

Therefore $m \geq n - k$ and $k \geq n - m$.

2. (20 points) Prove that if a tree has a node of degree d, then it has at least d leaves.

Solution: ince we are in a tree, any two vertices are connected by a unique path. Suppose that v has d neighbors $v_1, \ldots, v_d$. Any vertex u of distance n from v has exactly one neighbor which is distance $(n - 1)$ from v (along the path from v to u) and all other neighbors have distance $(n + 1)$ from v.

Consider d paths in the tree which are constructed as follows. We start from v, then go to v_i, then go to an arbitrary neighbor of v_i which is distance 2 from v, then go to an arbitrary neighbor of that vertex which is distance 3 from v and so on. This process will eventualy stop in a leaf, and all these leaves are different for different i. Therefore, there are at least d different leaves.

3. (20 points) A rooted binary tree has height n, that is, the distance from every vertex to the root is at most n. What is the maximal number of vertices in this tree?

Solution: We have 1 root (distance 0 from the root), 2 vertices of distance 1 from the root, 2^2 vertices of distance 2 from the root,...2^n

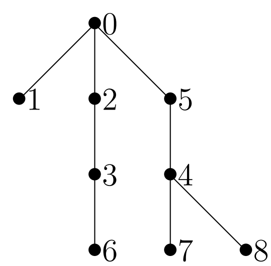
vertices of distance n from the root. In total there are $1+2+\ldots+2^n = 2^{n+1}-1$ vertices.

4. (20 points) A tree with 9 vertices (labeled from 0 to 8) has second row in Prüfer code 03204450. Reconstruct the first row in Prüfer code and the tree.

Solution: By Prüfer's algorithm we reconstruct the first row in the code:

1	6	3	2	7	8	4	5
0	3	2	0	4	4	5	0

and the tree



5. (20 points) Prove that the vertices in a tree can be colored black and white such that no two vertices of the same color are connected by an edge.

Solution 1: By a theorem from lecture, a graph can be colored in black and white if and only if it does not have a cycle of odd length. Since there are no cycles in a tree, it can be colores.

Solution 2: We prove by induction by the number of vertices n. For $n = 1$ it is clear. Suppose that we proved it for any tree with n vertices, consider a tree T with $(n+1)$ vertices. Since T is a tree, it has a leaf v. Then $T - v$ is a tree with n vertices, and can be colored by the assumption of induction. If the neigbor of v in $T - v$ is colored white, we color v black, if the neigbor of v in $T - v$ is colored black, we color v white. Since v is a leaf, it has no other neighbors and it is a valid coloring.