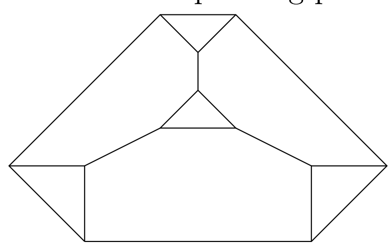
# MAT 145, Spring 2020
# Solutions to homework 8

**1.** (20 points) Consider the polyhedron obtained by truncating (cutting off) all vertices of a regular tetrahedron. How many vertices, edges and faces does it have? Draw the corresponding planar graph.

**Solution:** There are $3 \times 4 = 12$ vertices, $6 + 3 \times 4 = 18$ edges and 8 faces (4 triagles ans 4 hexagons). We can check the Euler formula: $12 - 18 + 8 = 2$. Here is the corresponding planar graph:



**2.** (20 points) A polyhedron has 12 faces, each has the shape of rhombus.
(a) How many edges does it have?
(b) How many vertices does it have? *Hint: use Euler formula*

**Solution:** Each face has 4 sides, and the sum of number of sides over all faces equals $2e$. Therefore $2e = 12 \cdot 4 = 48$ and $e = 24$. By Euler formula we get $v - 24 + 12 = 2$, so $v = 14$.

**3.** (20 points) Prove that in a planar bipartite graph one has $e \geq 2f$. *Hint: prove that* $2e \geq 4f$

**Solution:** Recall that in a planar graph each face has at least 3 sides. Since it is bipartite, it cannot have odd cycles, and all faces have at least 4 sides. Therefore $2e = \sum_{\text{faces}}(\text{number of sides}) \geq 4f$ and $e \geq 2f$.

**4.** (20 points) The graph $K_{3,3}$ has 3 black vertices and 3 white vertices, and every black vertex is connected with every white vertex. Use Problem 3 to prove that $K_{3,3}$ is not planar.

**Solution:** Assume that it is planar. This graph has 6 vertices and 9 edges, so by Euler formula we get $6 - 9 + f = 2$, so $f = 5$. By Problem 3, we get $e = 9 \geq 2f = 10$, contradiction. Therefore $K_{3,3}$ is not planar.

**5.** (20 points) The seams on a soccer ball form a graph. It has 60 vertices, each of degree 3. All faces are either pentagons or hexagons.
(a) How many edges does it have?
(b) How many faces in total does it have?

(c) How many pentagons are among the faces?

**Solution:** The total sum of degrees of vertices equals $2e$, so $2e = 60 \cdot 3 = 180$, and $e = 90$. By Euler formula we get $60 - 90 + f = 2$, so $f = 32$. Assume that there are $x$ pentagons and $y$ hexagons. We get two equations for $x$ and $y$: first, $x + y = f = 32$. Second, $2e = \sum_{\text{faces}}(\text{number of sides})$, so $180 = 5x + 6y$. We get:

$$x + y = 32,\ 5x + 6y = 180 \Rightarrow 5x + 5y = 5 \cdot 32 = 160 \Rightarrow y = 20, x = 12.$$

Therefore there are 12 pentagons and 20 hexagons.