Note that this practice sheet contains more problems than the actual midterm

1. Let $b_n$ be the number of ways to change $n$ cents using coins valued 4 and 6 cents. Find the generating function $\sum b_n x^n$. For which $n$ we can actually change $n$ cents this way (that is, $b_n > 0$)?

2. A permutation $f$ has order 3 if for all $x$ one has $f(f(f(x))) = x$ and $f$ is not the identity permutation. Find the exponential generating function for the number of permutations of order 3.

3. The generating function for the sequence $a_n$ equals $\sum a_n x^n = A(x)$. Find the generating function for the sequence $b_n = 5a_{n+1} + a_{n-1} - 3$.

4. Consider the series

$$A(x) = \sum a_n x^n = \frac{(1 + x)(1 + 2x)}{(1 - 2x)(1 + x^2)}.$$

(a) Find the poles of $A(x)$ and principal parts at these poles.

(b) Find the radius of convergence for $A(x)$.

(c) Use (b) to estimate the coefficients $a_n$.

(d) Use (a) to find the partial fraction decomposition of $A(x)$

(e) Find the closed formula for $a_n$.

5. The sequence $a_n$ satisfies the recurrence relation $a_n = a_{n-1} + 2a_{n-2}, a_0 = a_1 = 1$. Find the generating function $\sum a_n x^n$.

6. The exponential generating function $A(x) = \sum \frac{a_n}{n!} x^n$ satisfies the differential equation

$$A'(x) + A(x) = e^x, \quad A(0) = 0.$$

(a) Find the closed formula for the coefficients $a_n$.

(b) Find the closed formula for the series $A(x)$.

7. Find the closed formula for the coefficients of the series:

(a) $A(x) = \frac{1 + x}{(1 + 2x)(1 + 3x)}.$
(b) \( A(x) = \frac{1+x}{(1-x)^2} \).

8. Find the generating function for the number of partitions \textbf{without} parts equal to 3 or 5.

9. Use the exponential formula to find the number of set partitions \textbf{without} blocks of size 3 or 5.

10*. The \textit{Bessel function} \( A(x) = \sum_{n=0}^{\infty} a_n x^n \) satisfies the differential equation

\[
x A''(x) + A'(x) + xA(x) = 0.
\]

Find a closed formula for its coefficients.