MAT 146, Spring 2019
Solutions to homework 1

Section 1.7: 1. (40 points) Find the generating functions for each of the following sequences in closed form. In each case the sequence is defined for $n \geq 0$: (a) $a_n = n$ (b) $a_n = \alpha n + \beta$ (c) $a_n = n^2$ (f) $a_n = 3^n$ (g) $a_n = 5 \cdot 7^n - 3 \cdot 4^n$.

Solution: (a) (5 points)

$$\sum_{n=0}^{\infty} nz^n = (z\frac{d}{dz})\sum_{n=0}^{\infty} z^n = (z\frac{d}{dz})\frac{1}{1-z} = \frac{z}{(1-z)^2}.$$

(b) (10 points)

$$\sum_{n=0}^{\infty}(\alpha n + \beta)z^n = \alpha\sum_{n=0}^{\infty} nz^n + \beta\sum_{n=0}^{\infty} z^n =$$

$$\frac{\alpha z}{(1-z)^2} + \frac{\beta}{(1-z)} = \frac{\alpha z + \beta - \beta z}{(1-z)^2} = \frac{(\alpha - \beta)z + \beta}{(1-z)^2}.$$

(c) (10 points)

$$\sum_{n=0}^{\infty} n^2 z^n = (z\frac{d}{dz})\sum_{n=0}^{\infty} nz^n = (z\frac{d}{dz})\frac{z}{(1-z)^2} =$$

$$z \cdot \frac{1(1-z)^2 - z \cdot 2(1-z)\cdot(-1)}{(1-z)^4} = z \cdot \frac{(1-z) + 2z}{(1-z)^3} = \frac{z(1+z)}{(1-z)^3}.$$

(f) (5 points)

$$\sum_{n=0}^{\infty} 3^n z^n = \sum_{n=0}^{\infty}(3z)^n = \frac{1}{1-3z}$$

(g) (10 points)

$$\sum_{n=0}^{\infty}(5\cdot 7^n - 3\cdot 4^n)z^n = 5\sum_{n=0}^{\infty}(7z)^n - 3\sum_{n=0}^{\infty}(4z)^n = \frac{5}{1-7z} - \frac{3}{1-4z}.$$

3. (30 points) If $f(x)$ is the generating function of the sequence $\{a_n\}_{n\geq 0}$, then express simply, in terms of $f(x)$, the generating functions of the following sequences: (a) $\{a_n + c\}$ (b) $\{\alpha a_n + c\}$ (c) $\{na_n\}$ (e) $0, a_1, a_2, a_3, \ldots$ (g) $a_0, 0, a_2, 0, a_4, 0, a_6, 0, a_8, \ldots$ (h) $a_1, a_2, a_3, \ldots$

Solution: (a) (5 points)

$$\sum_{n=0}^{\infty}(a_n + c)x^n = \sum_{n=0}^{\infty} a_n x^n + c\sum_{n=0}^{\infty} x^n = f(x) + \frac{c}{1-x}.$$

(b) (5 points)

$$\sum_{n=0}^{\infty}(\alpha a_n + c)x^n = \alpha\sum_{n=0}^{\infty} a_n x^n + c\sum_{n=0}^{\infty} x^n = \alpha f(x) + \frac{c}{1-x}.$$

(c) (5 points)

$$\sum_{n=0}^{\infty} n a_n x^n = (x\frac{d}{dx})\sum_{n=0}^{\infty} a_n x^n = xf'(x).$$

(e) (5 points) $f(x) - a_0$
(g) (5 points) $f(-x) = a_0 - a_1 x + a_2 x^2 - \ldots$, so

$$\frac{1}{2}(f(x) + f(-x)) = a_0 + a_2 x^2 + a_4 x^4 + \ldots.$$

(h) (5 points) $(f(x) - a_0)/x$.

6. (30 points) In each part, a sequence $\{a_n\}_{n\geq 0}$ satisfies the given recurrence relation. Find the generating function for this sequence:

$$\text{(a)} \qquad a_{n+1} = 3a_n + 2 \qquad (n \geq 0, a_0 = 0),$$
$$\text{(b)} \qquad a_{n+2} = 2a_{n+1} - a_n \qquad (n \geq 0, a_0 = 0, a_1 = 1),$$
$$\text{(c)} \qquad a_{n+1} = a_n/3 + 1 \qquad (n \geq 0, a_0 = 0).$$

Solution: (a) (10 points) Let $A(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + \sum_{n=0}^{\infty} a_{n+1}x^{n+1}$, then $xA(x) = \sum_{n=0}^{\infty} a_n x^{n+1}$. Therefore

$$A(x) = a_0 + \sum_{n=0}^{\infty} a_{n+1}x^{n+1} = a_0 + 3\sum_{n=0}^{\infty} a_n x^{n+1} + \sum_{n=0}^{\infty} x^{n+1} = a_0 + 3xA(x) + \frac{x}{1-x},$$

and since $a_0 = 0$, we get

$$(1-3x)A(x) = \frac{x}{1-x}, \quad A(x) = \frac{x}{(1-x)(1-3x)}.$$

(b)(10 points) Let

$$A(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + \sum_{n=0}^{\infty} a_{n+1}x^{n+1} = a_0 + a_1 x + \sum_{n=0}^{\infty} a_{n+2}x^{n+2},$$

then

$$xA(x) = a_0 x + \sum_{n=0}^{\infty} a_{n+1}x^{n+2}, \quad x^2 A(x) = \sum_{n=0}^{\infty} a_n x^{n+2}.$$

Therefore

$$A(x) - 2xA(x) + x^2 A(x) = a_0 + a_1 x + \sum_{n=0}^{\infty} a_{n+2}x^{n+2} - 2a_0 x - 2\sum_{n=0}^{\infty} a_{n+1}x^{n+2} + \sum_{n=0}^{\infty} a_n x^{n+2} =$$

$$a_0 + (a_1 - 2a_0)x + \sum_{n=0}^{\infty}(a_{n+2} - 2a_{n+1} + a_n)x^n = a_0 + (a_1 - 2a_0)x.$$

Since $a_0 = 0, a_1 = 1$, we get

$$A(x)(1 - 2x + x^2) = x, \quad A(x) = \frac{x}{(1-x)^2}$$

Remark: This generating function agrees with the one for $a_n = n$ above. Indeed, $(n+2) = 2(n+1) - n$ and the initial conditions are satisfied.

(c) (10 points) Let $A(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + \sum_{n=0}^{\infty} a_{n+1}x^{n+1}$, then $xA(x) = \sum_{n=0}^{\infty} a_n x^{n+1}$. Therefore

$$A(x) - \frac{1}{3}xA(x) = a_0 + \sum_{n=0} a_{n+1}x^{n+1} - \frac{1}{3}\sum_{n=0}^{\infty} a_n x^{n+1}. = a_0 + \sum_{n=0}^{\infty} x^{n+1} = a_0 + \frac{x}{1-x}.$$

Since $a_0 = 0$, we get

$$A(x)(1 - \frac{1}{3}x) = \frac{x}{1-x}, \quad A(x) = \frac{x}{(1-\frac{1}{3}x)(1-x)} = \frac{3x}{(3-x)(1-x)}.$$