MAT 146, Spring 2019
Solutions to homework 2

A. (25 points) (a) Find a closed formula for the generating function $\sum_{k=0}^{\infty} k^3 z^k$.
(b) Use the result of (a) to find a closed formula for the sum of cubes:

$$P_3(N) = 1^3 + \ldots + N^3.$$

Solution: (a) (15 points) We have

$$\frac{1}{(1-z)^4} = \sum_{k=0}^{\infty} \frac{1}{6}(k+3)(k+2)(k+1)z^k,$$

$$\frac{1}{(1-z)^3} = \sum_{k=0}^{\infty} \frac{1}{2}(k+2)(k+1)z^k,$$

$$\frac{1}{(1-z)^2} = \sum_{k=0}^{\infty} (k+1)z^k,$$

$$\frac{1}{(1-z)} = \sum_{k=0}^{\infty} z^k.$$

Since

$$(k+3)(k+2)(k+1) = k^3 + 6k^2 + 11k + 6,\ (k+2)(k+1) = k^2 + 3k + 2,$$

we have

$$k^3 = (k^3 + 6k^2 + 11k + 6) - 6(k^2 + 3k + 2) + 7(k+1) - 1 =$$

$$6\frac{1}{6}(k+3)(k+2)(k+1) - 12\frac{1}{2}(k+2)(k+1) + 7(k+1) - 1,$$

and

$$\sum_{k=0}^{\infty} k^3 z^k = \frac{6}{(1-z)^4} - \frac{12}{(1-z)^3} + \frac{7}{(1-z)^2} - \frac{1}{(1-z)}. \qquad (1)$$

One can simplify it using common denominator:

$$\sum_{k=0}^{\infty} k^3 z^k = \frac{6 - 12(1-z) + 7(1-z)^2 - (1-z)^3}{(1-z)^4} = \frac{z^3 + 4z^2 + z}{(1-z)^4}.$$

(b) (10 points) The sum $P_3(N) = 1^3 + \ldots + N^3$ is the coefficient at z^N in the series

$$\left(\sum_{k=0}^{\infty} k^3 z^k\right)\left(\sum_{k=0}^{\infty} z^k\right) = \frac{1}{(1-z)}\left(\sum_{k=0}^{\infty} k^3 z^k\right) =$$

(by (1))

$$= \frac{6}{(1-z)^5} - \frac{12}{(1-z)^4} + \frac{7}{(1-z)^3} - \frac{1}{(1-z)^2}.$$

Therefore

$$P_3(N) = 6 \cdot \frac{1}{24}(N+4)(N+3)(N+2)(N+1) - 12 \cdot \frac{1}{6}(N+3)(N+2)(N+1) +$$

$$7\frac{1}{2}(N+2)(N+1) - (N+1) =$$

$$\frac{N+1}{4}((N+4)(N+3)(N+2) - 8(N+3)(N+2) + 14(N+2) - 4) =$$

$$\frac{N+1}{4}\left(N^3 + 9N^2 + 26N + 24 - 8N^2 - 40N - 48 + 14N + 28 - 4\right) =$$

$$\frac{N+1}{4}(N^3 + N^2) = \frac{1}{4}N^2(N+1)^2.$$

See also more combinatorial proofs of this fact at:

`https://en.wikipedia.org/wiki/Squared_triangular_number`

Section 1.7: 11. (25 points) Let $f(n)$ be the number of subsets of $\{1, \ldots, n\}$ that contain no consecutive numbers. Find the recurrence that is satisfied by $f(n)$ and then "find" the numbers themselves.

Solution: Let A be a subset of $\{1, \ldots, n\}$ with no consecutive numbers. If A does not contain n, then A is a subset of $\{1, \ldots, n-1\}$ and there are $f(n-1)$ choices for it. If A contains n then it cannot contain $n-1$, and the rest of A is an arbitrary subset of $\{1, \ldots, n-2\}$ with no consecutive integers, there are $n-2$ choices for it. Therefore

$$f(n) = f(n-1) + f(n-2).$$

If $n = 1$ we get two such subsets $\emptyset$ and $\{1\}$, if $n = 2$ there are 3 such subsets $\emptyset$, $\{1\}$, $\{2\}$. Therefore $f(1) = 2$ and $f(2) = 3$. Recall that Fibonacci numbers F_n are given by the same recursion $F_n = F_{n-1} + F_{n-2}$, but $F_0 = F_1 = 1, F_2 = 2, F_3 = 3$. Therefore $f(n) = F_{n+1}$.

12. (25 points) Let $f(n,k)$ be the number of k-element subsets of $\{1,\ldots,n\}$ that contain no consecutive numbers. Find the recurrence for $f(n,k)$, solve it and find the formula for $f(n,k)$. Show the numerical valued of $f(n,k)$ in a Pascal triangle arrangement for $n \le 6$.

Solution: Similarly to the previous problem we get a recursion

$$f(n,k) = f(n-1,k) + f(n-2,k-1).$$

To solve it, it is useful to plot $f(n,k)$ for small n and k in a table

n	k=0	k=1	k=2	k=3
1	1	1	0	0
2	1	2	0	0
3	1	3	1	0
4	1	4	3	0
5	1	5	6	1

This table is very similar to the Pascal triangle containg binomial coefficients:

$$\begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & 1 & & 1 & & & \\ & & 1 & & 2 & & 1 & & \\ & 1 & & 3 & & 3 & & 1 & \\ 1 & & 4 & & 6 & & 4 & & 1 \end{array}$$

So one can guess

$$f(n,k) = \binom{n-k+1}{k}.$$

Let us prove this equation by induction in n. If $n = 1$ we get

$$f(1,0) = 1 = \binom{1-0+1}{0},\ f(1,1) = 1 = \binom{1-1+1}{1}.$$

If $n = 2$ we get

$$f(2,0) = 1 = \binom{2-0+1}{0},\ f(2,1) = 2 = \binom{2-1+1}{1},\ f(2,2) = 0 = \binom{2-2+1}{2}.$$

Now for $n > 2$ we get

$$f(n-1,k) + f(n-2,k-1) = \binom{n-1-k+1}{k} + \binom{n-2-(k-1)+1}{k-1} =$$

$$\binom{n-k}{k} + \binom{n-k}{k-1} = \binom{n-k+1}{k}.$$

13. (25 points) By comparing the results of the above two exercises, deduce an identity. Draw a picture of the elements of Pascal triangle that are involved in this identity.

Solution: We have $f(n) = \sum_{k=0}^{n} f(n,k)$, so

$$f(n) = F_{n+1} = \sum_k \binom{n-k+1}{k}.$$

We can change $n+1$ to n and get

$$F_n = \sum_k \binom{n-k}{k}.$$