MAT 146, Spring 2019
Solutions to homework 3

A. (a) (20 points) Use Taylor formula to find the coefficients in the series $A(x) = \sqrt{1-x}$.

Solution: We have $A(x) = (1-x)^{1/2}$, so $A'(x) = -\frac{1}{2}(1-x)^{-1/2}$, $A''(x) = -\frac{1}{2}\cdot\frac{1}{2}(1-x)^{-3/2}$, $A'''(x) = -\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{3}{2}(1-x)^{-5/2}$, and in general for $n \geq 2$

$$A^{(n)}(x) = -\frac{1\cdot 3\cdots(2n-3)}{2^n}(1-x)^{-(2n-1)/2},$$

$$A^{(n)}(0) = -\frac{1\cdot 3\cdots(2n-3)}{2^n} = -\frac{(2n-2)!}{2^n\cdot 2^{n-1}(n-1)!}.$$

Note that for $n = 1$ we have $(2n-2)! = (n-1)! = 1$, and the formula for $A^{(n)}(0)$ holds for $n = 1$. Therefore by Taylor formula

$$A(x) = 1 - \sum_{n=1}^{\infty} \frac{(2n-2)!}{2^{2n-1}(n-1)!n!}x^n.$$

(b) (20 points) We proved in class that the generating function for Catalan numbers has the form:

$$C(x) = \sum_{n=0}^{\infty} c_n x^n = \frac{1-\sqrt{1-4x}}{2x}.$$

Use the result of part (a) to get an explicit formula for c_n.

Solution: From part (a) we get

$$\sqrt{1-4x} = 1 - \sum_{n=1}^{\infty} \frac{(2n-2)!}{2^{2n-1}(n-1)!n!}(4x)^n = 1 - \sum_{n=1}^{\infty} \frac{(2n-2)!\cdot 2}{(n-1)!n!}x^n.$$

Now

$$C(x) = \frac{1-\sqrt{1-4x}}{2x} = \sum_{n=1}^{\infty} \frac{(2n-2)!\cdot 2}{(n-1)!n!}\frac{x^n}{2x} = \sum_{n=1}^{\infty} \frac{(2n-2)!}{(n-1)!n!}x^{n-1} = \sum_{n=0}^{\infty} \frac{(2n)!}{n!(n+1)!}x^n$$

and

$$c_n = \frac{(2n)!}{n!(n+1)!} = \frac{1}{n+1}\binom{2n}{n}.$$

B. (a) (20 points) Find the generation function $\sum_{n=1}^{\infty} \frac{x^n}{n}$.

Solution: Let $A(x) = \sum_{n=1}^{\infty} \frac{x^n}{n}$, then

$$A'(x) = \sum_{n=1}^{\infty} x^{n-1} = \sum_{n=0}^{\infty} x^n = \frac{1}{1-x},$$

and $A(x) = \int \frac{dx}{1-x} = -\ln(1-x) + C$. At $x = 0$ we get $A(0) = 0$, so $C = 0$ and $A(x) = -\ln(1-x)$.

(b) (20 points) The harmonic numbers H_n are defined as

$$H_n = 1 + \frac{1}{2} + \ldots + \frac{1}{n}.$$

Find the generating function $\sum H_n x^n$.

Solution: We have

$$\sum H_n x^n = (\sum_{n=1}^{\infty} \frac{x^n}{n})(\sum_{n=0}^{\infty} x^n) = -\ln(1-x) \cdot \frac{1}{1-x}.$$

C. (20 points) Use generating functions to prove the identity

$$F_0 + \ldots + F_n = F_{n+2} - 1$$

where F_k are Fibonacci numbers.

Solution: Let $F(x) = \sum_n F_n x^n$ be the generating function for Fibonacci numbers. We proved in class that $F(x) = \frac{1}{1-x-x^2}$. Now the generating function for $F_0 + \ldots + F_n$ equals

$$F(x) \cdot \frac{1}{1-x} = \frac{1}{(1-x-x^2)(1-x)}$$

while the generating function for $F_{n+2} - 1$ equals

$$\frac{F(x) - F_0 - F_1 x}{x^2} - \frac{1}{1-x} = \frac{\frac{1}{1-x-x^2} - 1 - x}{x^2} - \frac{1}{1-x} =$$

$$\frac{1 - 1 - x + x + x^2 + x^2 + x^3}{(1-x-x^2)x^2} - \frac{1}{1-x} = \frac{2+x}{1-x-x^2} - \frac{1}{1-x} =$$

$$\frac{2 + x - 2x - x^2 - 1 + x + x^2}{(1-x-x^2)(1-x)} = \frac{1}{(1-x-x^2)(1-x)}.$$

Since the generating functions agree, we have

$$F_0 + \ldots + F_n = F_{n+2} - 1.$$