MAT 146, Spring 2019
Solutions to homework 4

Section 1.7: 4. (30 points) Let $f(x)$ be the **exponential** generating function of a sequence $\{a_n\}$. Find the exponential generating functions for the following sequences in terms of $f(x)$: (a) (5 points) $\{a_n + c\}$; (b) (5 points) $\{\alpha a_n + c\}$ (c) (5 points) $\{na_n\}$; (e) (5 points) $0, a_1, a_2, a_3, \ldots$; (g) (5 points) $a_0, 0, a_2, 0, a_4, 0, \ldots$; (h) (5 points) $a_1, a_2, a_3, \ldots$

Solution: (a) $\sum \frac{a_n+c}{n!}x^n = f(x) + ce^x$; (b) $\sum \frac{\alpha a_n+c}{n!}x^n = \alpha f(x) + ce^x$. (c) $\sum \frac{na_n}{n!}x^n = xf'(x)$ (d) $f(x) - a_0$ (g) $\frac{1}{2}(f(x) + f(-x))$ (h) $f'(x)$.

8. (a) (20 points) A sequence $\{a_n\}$ satisfies the recurrence relation $a_{n+1} = 3a_n + 2,\ a_0 = 0$. Find the exponential generating function $\sum_{n=0}^{\infty} \frac{a_n}{n!}x^n$.

Solution 1: Let $\sum_{n=0}^{\infty} \frac{a_n}{n!}x^n = A(x)$, then $\sum \frac{a_{n+1}x^n}{n!} = A'(x)$. We get a differential equation $A'(x) = 3A(x) + 2e^x$. Observe that $A(x) = -e^x$ satisfies this equation, so the general solution has the form

$$A(x) = -e^x + Ce^{3x}.$$

Since $a_0 = A(0) = 0$, we have $-1 + C = 0, C = 1$ and $A(x) = -e^x + e^{3x}$.

Solution 2: We can solve the recurrence using usual generating functions first. Let $B(x) = \sum a_n x^n$, then $\sum a_{n+1}x^n = (B(x) - a_0)/x = B(x)/x$. We have

$$B(x)/x = 3B(x) + \frac{2}{1-x},\ B(x) = 3xB(x) + \frac{2x}{1-x},\ B(x)(1-3x) = \frac{2x}{1-x}$$

and

$$B(x) = \frac{2x}{(1-x)(1-3x)} = \frac{1}{1-3x} - \frac{1}{1-x} = \sum_n (3^n - 1)x^n.$$

Therefore $a_n = 3^n - 1$ and

$$A(x) = \sum_{n=0}^{\infty} \frac{a_n}{n!}x^n = \sum_n \frac{(3^n-1)x^n}{n!} = e^{3x} - e^x.$$

Section 2.7: 20. (25 points) Prove the binomial theorem

$$(x+y)^n = \sum_k \binom{n}{k} x^k y^{n-k}$$

by comparing the coefficient of $t^n/n!$ on both sides of the equation $e^{t(x+y)} = e^{tx}e^{ty}$. Prove the multinomial theorem

$$(x_1 + \ldots + x_k)^n = \sum_{r_1+\ldots+r_k=n} \frac{n!}{r_1! \cdots r_k!} x_1^{r_1} \cdots x_k^{r_k}$$

by a similar method.

Solution: We have

$$\sum_n \frac{1}{n!}(x_1 + \ldots + x_k)^n t^n = e^{t(x_1+\ldots+x_k)} = e^{tx_1} \cdots e^{tx_k} = \sum_{r_1} \frac{x_1^{r_1}}{r_1!} \cdots \sum_{r_k} \frac{x_k^{r_k}}{r_k!},$$

so

$$\frac{1}{n!}(x_1 + \ldots + x_k)^n = \sum_{r_1+\ldots+r_k=n} \frac{1}{r_1! \cdots r_k!} x_1^{r_1} \cdots x_k^{r_k}.$$

By multiplying by $n!$ we get the desired identity.

27. (25 points) Let $D(n)$ be the number of derangements on n letters. We proved in class that the exponential generating function for $D(n)$ has the form

$$D(x) = \frac{e^{-x}}{1-x} = \sum_n \frac{D(n)}{n!} x^n.$$

(b) Prove, by any method, that $D(n+1) = (n+1)D(n) + (-1)^{n+1}$

Solution: We have

$$\sum_n \frac{D(n+1)}{n!} x^n = D'(x) = \frac{-e^{-x}(1-x) - e^{-x}(-1)}{(1-x)^2} = \frac{xe^{-x}}{(1-x)^2}.$$

On the other hand,

$$\sum_n ((n+1)D(x) + (-1)^{n+1}) \frac{x^n}{n!} = xD'(x) + D(x) - e^{-x} = \frac{x^2 e^{-x}}{(1-x)^2} + \frac{e^{-x}}{1-x} - e^{-x} =$$

$$\frac{e^{-x}(x^2 + 1 - x - (1-x)^2)}{(1-x)^2} = \frac{e^{-x}(x^2 + 1 - x - 1 + 2x - x^2)}{(1-x)^2} = \frac{xe^{-x}}{(1-x)^2}.$$

Therefore the exponential generating functions for the left and right hand sides coincide, and the recursion holds.

(c) Prove, by any method, that $D(n+1) = n(D(n) + D(n-1))$

Solution: We have

$$D(n+1) = (n+1)D(n) + (-1)^{n+1},\ D(n) = nD(n-1) + (-1)^n,$$

so we can add these equations and get

$$D(n+1)+D(n) = (n+1)D(n)+nD(n-1),\ D(n+1) = nD(n)+nD(n-1).$$

(e) Let $D_k(n)$ be the number of permutations of n letters with exactly k fixed points. Show that

$$\sum_{k,n} D_k(n)\frac{x^n y^k}{n!} = \frac{e^{-x(1-y)}}{1-x}.$$

Solution: We have $D_k(n) = \binom{n}{k}D(n-k)$, so

$$\sum_{k,n} D_k(n)\frac{x^n y^k}{n!} = \sum_{k,n} \binom{n}{k} D(n-k)\frac{x^n y^k}{n!} = \sum_{n,k} \frac{n!}{k!(n-k)!} D(n-k)\frac{x^n y^k}{n!} =$$

$$\sum_{n,k} \frac{x^k y^k}{k!} \cdot \frac{x^{n-k}D(n-k)}{(n-k)!} = e^{xy} \cdot \frac{e^{-x}}{1-x} = \frac{e^{xy-x}}{1-x}.$$