Section 1.7: 4. (30 points) Let $f(x)$ be the exponential generating function of a sequence $\{a_n\}$. Find the exponential generating functions for the following sequences in terms of $f(x)$: (a) (5 points) $\{a_n + c\}$; (b) (5 points) $\{\alpha a_n + c\}$; (c) (5 points) $\{na_n\}$; (e) (5 points) $0, a_1, a_2, a_3, \ldots$; (g) (5 points) $a_0, a_2, 0, a_4, 0, \ldots$; (h) (5 points) $a_1, a_2, a_3, \ldots$

Solution: (a) $\sum a_n + c n! x^n = f(x) + c e^x$; (b) $\sum \alpha a_n + c n! x^n = \alpha f(x) + c e^x$. (c) $\sum n a_n n! x^n = x f'(x)$ (d) $f(x) - a_0 (g) \frac{1}{2} (f(x) + f(-x))$ (h) $f'(x)$.

8. (30 points) A sequence $\{a_n\}$ satisfies the recurrence relation $a_{n+1} = 3a_n + 2$, $a_0 = 0$. Find the exponential generating function $\sum_{n=0}^{\infty} \frac{a_n}{n!} x^n$.

Solution 1: Let $\sum_{n=0}^{\infty} \frac{a_n}{n!} x^n = A(x)$, then $\sum_{n=0}^{\infty} \frac{a_{n+1} x^n}{n!} = A'(x)$. We get a differential equation $A'(x) = 3A(x) + 2e^x$. Observe that $A(x) = -e^x$ satisfies this equation, so the general solution has the form

$$A(x) = -e^x + C e^{3x}.$$ 

Since $a_0 = A(0) = 0$, we have $-1 + C = 0$, $C = 1$ and $A(x) = -e^x + e^{3x}$.

Solution 2: We can solve the recurrence using usual generating functions first. Let $B(x) = \sum a_n x^n$, then $\sum a_{n+1} x^n = (B(x) - a_0)/x = B(x)/x$. We have

$$B(x)/x = 3B(x) + \frac{2}{1 - x}, \quad B(x) = 3xB(x) + \frac{2x}{1 - x}, \quad B(x)(1 - 3x) = \frac{2x}{1 - x}$$

and

$$B(x) = \frac{2x}{(1 - x)(1 - 3x)} = \frac{1}{1 - 3x} - \frac{1}{1 - x} = \sum_n (3^n - 1) x^n.$$ 

Therefore $a_n = 3^n - 1$ and

$$A(x) = \sum_{n=0}^{\infty} \frac{a_n}{n!} x^n = \sum_n \frac{(3^n - 1) x^n}{n!} = e^{3x} - e^x.$$
Section 2.7: 20. (25 points) Prove the binomial theorem

$$(x + y)^n = \sum_k \binom{n}{k} x^k y^{n-k}$$

by comparing the coefficient of $t^n/n!$ on both sides of the equation $e^{t(x+y)} = e^{tx}e^{ty}$. Prove the multinomial theorem

$$(x_1 + \ldots + x_k)^n = \sum_{r_1 + \ldots + r_k = n} \frac{n!}{r_1! \cdots r_k!} x_1^{r_1} \cdots x_k^{r_k}$$

by a similar method.

Solution: We have

$$\sum_n \frac{1}{n!} (x_1 + \ldots + x_k)^n t^n = e^{t(x_1 + \ldots + x_k)} = e^{tx_1} \cdots e^{tx_k} = \sum_{r_1} \frac{x_1^{r_1}}{r_1!} \cdots \sum_{r_k} \frac{x_k^{r_k}}{r_k!},$$

so

$$\frac{1}{n!} (x_1 + \ldots + x_k)^n = \sum_{r_1 + \ldots + r_k = n} \frac{1}{r_1! \cdots r_k!} x_1^{r_1} \cdots x_k^{r_k}.$$  

By multiplying by $n!$ we get the desired identity.

27. (25 points) Let $D(n)$ be the number of derangements on $n$ letters. We proved in class that the exponential generating function for $D(n)$ has the form

$$D(x) = \frac{e^{-x}}{1-x} = \sum_n \frac{D(n)}{n!} x^n.$$  

(b) Prove, by any method, that $D(n+1) = (n+1)D(n) + (-1)^{n+1}$

Solution: We have

$$\sum_n \frac{D(n+1)}{n!} x^n = D'(x) = \frac{-e^{-x}(1-x) - e^{-x}(-1)}{(1-x)^2} = \frac{xe^{-x}}{(1-x)^2}.$$  

On the other hand,

$$\sum_n ((n+1)D(x)+(-1)^{n+1}) \frac{x^n}{n!} = xD'(x)+D(x)e^{-x} = \frac{x^2e^{-x}}{(1-x)^2} + \frac{e^{-x}}{1-x} - e^{-x} = \frac{e^{-x}(x^2 + 1 - x - (1-x)^2)}{(1-x)^2} = \frac{e^{-x}(x^2 + 1 - x - 1 + 2x - x^2)}{(1-x)^2} = \frac{xe^{-x}}{(1-x)^2}.$$
Therefore the exponential generating functions for the left and right hand sides coincide, and the recursion holds.

(c) Prove, by any method, that $D(n+1) = n(D(n) + D(n-1))$

**Solution:** We have

$$D(n+1) = (n+1)D(n) + (-1)^{n+1}, \quad D(n) = nD(n-1) + (-1)^n,$$

so we can add these equations and get

$$D(n+1) + D(n) = (n+1)D(n) + nD(n-1), \quad D(n+1) = nD(n) + nD(n-1).$$

(e) Let $D_k(n)$ be the number of permutations of $n$ letters with exactly $k$ fixed points. Show that

$$\sum_{k,n} D_k(n) \frac{x^ny^k}{n!} = \frac{e^{-x(1-y)}}{1-x}.$$ 

**Solution:** We have $D_k(n) = \binom{n}{k} D(n-k)$, so

$$\sum_{k,n} D_k(n) \frac{x^ny^k}{n!} = \sum_{k,n} \binom{n}{k} D(n-k) \frac{x^ny^k}{n!} = \sum_{n,k} \frac{n!}{k!(n-k)!} D(n-k) \frac{x^ny^k}{n!} =$$

$$\sum_{n,k} \frac{x^ky^k}{k!} \cdot \frac{x^{n-k} D(n-k)}{(n-k)!} = e^{xy} \cdot \frac{e^{-x}}{1-x} = \frac{e^{xy}-x}{1-x}.$$ 

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