MAT 146, Spring 2019
Solutions to homework 5

Section 3.19: 2. (25 points) Let $f(n,k)$ be the number of permutations of n letters whose cycle lengths are all divisible by k. Find a simple explicit expression for the exponential generating function $\sum_n f(n,k)\frac{x^n}{n!}$.

Solution: We can use the exponential formula where the cards are cycles of length $k, 2k, \ldots$. There are $(mk-1)!$ cycles of length mk on mk elements, so the exponential generating function for the number of cars equals

$$D(x) = \sum_{m=1}^{\infty}(mk-1)!\frac{x^{mk}}{(mk)!} = \sum_{m=1}^{\infty}\frac{x^{mk}}{mk} = \frac{1}{k}\sum_{m=1}^{\infty}\frac{(x^k)^m}{m} =$$

$$-\frac{1}{k}\ln(1-x^k).$$

Then by the exponential formula we get

$$H(x) = e^{D(x)} = e^{-\frac{1}{k}\ln(1-x^k)} = \frac{1}{(1-x^k)^{1/k}}.$$

3. (25 points) Find the exponential generating function for the number of set partitions of $\{1,\ldots,n\}$ where all parts have prime number of elements.

Solution: Here the cards are the sets with prime number of elements. There is just one card for each prime p, so

$$D(x) = \sum_{p \text{ prime}}\frac{x^p}{p!}.$$

By the exponential formula, we get

$$H(x) = e^{D(x)} = e^{\sum_{p \text{ prime}}\frac{x^p}{p!}}.$$

5. (25 points) Let T_n be the number of involutions in S_n, we proved in class that

$$T(x) = \sum_n \frac{T_n}{n!}x^n = e^{x+x^2/2}.$$

(a) (10 points) Find a recurrence for T_n.
(b) (5 points) Compute $T_1, \dots, T_6$.
(c) (10 points) Give a combinatorial interpretation of the recurrence.

Solution: (a) We have $T'(x) = (1+x)T(x)$, so

$$T'(x) = \sum_n T_{n+1} \frac{x^n}{n!} = T(x) + xT(x) = \sum_n T_n \frac{x^n}{n!} + \sum_n T_{n-1} \frac{x^n}{(n-1)!}.$$

By comparing the coefficients at x^n, we get

$$\frac{T_{n+1}}{n!} = \frac{T_n}{n!} + \frac{T_{n-1}}{(n-1)!},$$

so $T_{n+1} = T_n + nT_{n-1}$.

(b) We have $T_1 = 1, T_2 = 2$, so

$$T_3 = T_2 + 2T_1 = 4,\ T_4 = T_3 + 3T_2 = 4 + 3 \cdot 2 = 10,$$

$$T_5 = T_4 + 4T_3 = 10 + 4 \cdot 4 = 26,\ T_6 = T_5 + 5T_4 = 26 + 5 \cdot 10 = 76.$$

(c) Let $f \in S_{n+1}$ be an involution, then either $f(n+1) = n+1$ or $f(n+1) = x$ and $f(x) = n+1$ for some $x \in \{1, \dots, n\}$. In the first case there are $T(n)$ such involution, in the second case there are n ways to choose x and T_{n-1} ways to choose an involution for a given x. In total, we get

$$T_{n+1} = T_n + nT_{n-1}.$$

6. (25 points) Find the exponential generating function for the number of permutations in S_n with no cycles of length ≤ 3. Your answer should not contain any infinite series.

Solution: In this case the cards are cycles of length greater than 4, so the exponential generating function for them has a form

$$D(x) = \sum_{n=4}^{\infty} (n-1)! \frac{x^n}{n!} = \sum_{n=4}^{\infty} \frac{x^n}{n} = \sum_{n=1}^{\infty} \frac{x^n}{n} - (x + \frac{x^2}{2} + \frac{x^3}{3}) =$$

$$-\ln(1-x) - (x + \frac{x^2}{2} + \frac{x^3}{3}).$$

Now by the exponential formula we get

$$H(x) = e^{D(x)} = e^{-\ln(1-x) - (x + \frac{x^2}{2} + \frac{x^3}{3})} = \frac{e^{-(x + \frac{x^2}{2} + \frac{x^3}{3})}}{(1-x)}.$$