MAT 146, Spring 2019
Solutions to homework 6

Section 3.19: 20. (25 points) Find the largest integer that is *not* of the form $6x + 10y + 15z$ where x, y, z are nonnegative integers. Prove that your answer is correct, that is, your number is no representable in this form, and that every larger integer is so representable.

Solution: Let us prove that every number greater than or equal to 30 is representable. Indeed,

$$30 = 10+10+10,\ 31 = 6+10+15,\ 32 = 6+6+10+10,\ 33 = 15+6+6+6,$$

$$34 = 6+6+6+6+10,\ 35 = 15+10+10.$$

Now, every number greater than or equal to 30 can be written as $n = 30 + 6q + r$ where $p \geq 0$ and $0 \leq r \leq 5$, and we just proved that we can change $30 + r$, so we can change n as well.

Finally, let us prove that we cannot change 29. Indeed, suppose that $29 = 6x + 10y + 15z$, then $z = 0$ or $z = 1$. If $z = 0$ then $29 = 6x + 10y$ which is impossible since the right hand side is even. If $z = 1$ then

$$29 = 6x + 10y + 15,\ 14 = 6x + 10z$$

and it is easy to see that this is impossible.

A. (25 points) Define a sequence a_n as follows: $a_n = 1$ if n cents can be changed using 3- and 4-cent coins only, and $a_n = 0$ otherwise. Find the closed formula for the generating function $\sum a_n x^n$.

Solution: We can change 0, 3,4 and $n \geq 6$ cents, so

$$\sum a_n x^n = 1 + x^3 + x^4 + \sum_{n=6}^{\infty} x^n = 1 + x^3 + x^4 + \frac{x^6}{1-x} = \frac{1 - x + x^3 - x^5 + x^6}{1-x}.$$

B. (25 points) Generalizing the previous problem, let p and q be two coprime integers. Let $a_n = 1$ if n cents can be changed using p- and q-cent coins only, and $a_n = 0$ otherwise. Find the closed formula for the generating function $\sum a_n x^n$.

Solution: Every integer n can be uniquely written in the form $n = ap+bq$ where $0 \leq a \leq q-1$. It can be changed if and only if $b \geq 0$. Therefore

$$\sum_{n=0}^{\infty} a_n x^n = \sum_{a=0}^{q-1} \sum_{b=0}^{\infty} x^{ap+bq} = \sum_{a=0}^{q-1} x^{ap} \sum_{b=0}^{\infty} x^{bq} =$$

$$\frac{1-x^{pq}}{1-x^p} \cdot \frac{1}{1-x^q} = \frac{(1-x^{pq})}{(1-x^p)(1-x^q)}.$$

One can check that for $p=3, q=4$ we recover the answer to problem A above.

C. (25 points) Find the generating function for the number of partitions where all parts are divisible by 7.

Solution: We can say that this is a money changing problem where the coins have values $7, 14, \ldots$, so the generating function has the form

$$\frac{1}{(1-x^7)(1-x^{14})\cdots} = \prod_{k=1}^{\infty} \frac{1}{1-x^{7k}}.$$