MAT 146, Spring 2019
Solutions to homework 7

Section 2.7: 11. (25 points) Find the radius of convergence of the following series:
(a) $\sum_{n=1}^{\infty} \frac{x^n}{n^2}$; (c) $1 + 5x^2 + 25x^4 + 125x^6 + \dots$

Solution: (a) We have $\lim_{n\to\infty} \sqrt[n]{n} = 1$, so

$$\lim_{n\to\infty} \sqrt[n]{\frac{1}{n^2}} = \frac{1}{\lim_{n\to\infty} \sqrt[n]{n^2}} = 1.$$

Therefore $R = \frac{1}{\limsup_{n\to\infty} \sqrt[n]{a_n}} = 1$.

(b) We have

$$1 + 5x^2 + 25x^4 + 125x^6 + \dots = \frac{1}{1-5x^2},$$

so it has poles at $x = \pm\frac{1}{\sqrt{5}}$. The radius of convergence equals the distance from the origin to the nearest pole, so it is equal to $R = \frac{1}{\sqrt{5}}$.

A: (25 points) (a) Find the coefficients of the power series

$$A(x) = \sum_{n=0}^{\infty} a_n x^n = \frac{1-x}{(1-2x)(1+3x)}.$$

Solution: We have

$$\frac{1-x}{(1-2x)(1+3x)} = \frac{A}{1-2x} + \frac{B}{1+3x},$$

$$A(1+3x) + B(1-2x) = 1 - x, A + B = 1, 3A - 2B = -1.$$

Since $2A + 2B = 2$, we can add the equations and get $5A = 1$,

$$A = \frac{1}{5},\ B = \frac{4}{5}.$$

Therefore

$$A(x) = \frac{1}{5} \cdot \frac{1}{1-2x} + \frac{4}{5} \cdot \frac{1}{1+3x} = \sum_{n=0}^{\infty} (\frac{1}{5} \cdot 2^n + \frac{4}{5} \cdot (-3)^n) x^n.$$

(b) Find the radius of convergence of this series.

Solution: The function $A(x)$ has poles at $x = 1/2$ and $x = -1/3$, and the radius of convergence equals the distance from the origin to the nearest pole. Therefore $R = 1/3$.

B: (25 points) The sequence $\{a_n\}$ satisfies the recurrence relation

$$15a_n - 5a_{n-1} + 3a_{n-2} - a_{n-3} = 0$$

with some initial conditions a_0, a_1, a_2.
(a) Find the radius of convergence of the series $\sum a_n x^n$.

Solution: Let $A(x) = \sum a_n x^n$, then the recurrence translates into linear equation

$$15A(x) - 5xA(x) + 3x^2 A(x) - x^3 A(x) = 15a_0 + (15a_1 - 5a_0)x + (15a_2 - 5a_1 + 3a_0)x^2.$$

Therefore

$$A(x) = \frac{15a_0 + (15a_1 - 5a_0)x + (15a_2 - 5a_1 + 3a_0)x^2}{15 - 5x + 3x^2 - x^3} =$$

$$\frac{15a_0 + (15a_1 - 5a_0)x + (15a_2 - 5a_1 + 3a_0)x^2}{(5 + x^2)(3 - x)}.$$

Assuming there are no cancellations, the function has poles at $x = 3$ and $x = \pm\sqrt{5}i$. Since $\sqrt{5} < 3$, the radius of convergence equals $R = \sqrt{5}$.

(b) Estimate a_n without solving the recurrence.

Solution: Since

$$\limsup_{n\to\infty} \sqrt[n]{a_n} = \frac{1}{R} = \frac{1}{\sqrt{5}},$$

for all ε we have

$$|a_n| < \left(\frac{1}{\sqrt{5}} + \varepsilon\right)^n$$

for all but finitely many n, and

$$|a_n| > \left(\frac{1}{\sqrt{5}} - \varepsilon\right)^n$$

for infinitely many n.

C: (25 points) Recall that the exponential generating function for the number of derangements equals

$$D(x) = \sum \frac{D_n}{n!} x^n = \frac{e^{-x}}{1-x}.$$

(a) Find all poles of $D(x)$ and principal parts at these poles.

Solution: The function $D(x)$ has a pole at $x = 1$ with the principal part $\frac{e^{-1}}{1-x}$.

(b) Use "pole removal" procedure to estimate D_n.

Solution: The function

$$D(x) - \frac{e^{-1}}{1-x} = \frac{e^{-x} - e^{-1}}{1-x}$$

has a removable singularity at $x = 1$ (by the above, this can be also checked by L'Hôpital rule), and hence no poles. The corresponding series

$$D(x) - \frac{e^{-1}}{1-x} = \sum_{n=0}^{\infty} \left(\frac{D_n}{n!} - e^{-1} \right) x^n$$

converges everywhere, so $R = \infty$. Therefore for all ε one has

$$\left| \frac{D_n}{n!} - e^{-1} \right| < \varepsilon^n$$

for all but finitely many n.