MAT 148, Winter 2016
Practice problems for the final exam

1. a) Write the generator matrix and the parity check matrix for the $(1, 2)$ Reed-Muller code.
b) Describe all the cosets and their syndromes. How many elements are there in each coset?
c) How many errors could this code correct?

2. a) Prove that the code with the generating matrix:

$$G = \begin{pmatrix} 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{pmatrix}$$

corrects one error, but does not correct two errors.
b) Is this a perfect single-error correcting code? If not, compute the number of messages which cannot be decoded by changing one bit.
c) Write a message which cannot be decoded by changing one bit.
d) Write a message which can be decoded in several different ways by changing at most two bits.

3. A satellite transmits a message of length $n = 100$, which may contain at most 2 errors.
a) Prove that at least 13 bits should be redundant.
b) Prove that there is a double-error correcting code with 18 redundant bits.

4. A code has 3 redundant bits.
a) Prove that it cannot correct two errors.
b) What are the possible dimensions for this code, if it corrects one error?

5. a) Build the parity check matrix for the [15,11] Hamming code.
b) Decode the message 000111000111000 (*Note: the answer depends on the order of columns!*).

6. For all Reed-Muller codes of length 64, compute their dimensions and number of errors they correct.

7. Simplify the following expressions in $\mathbb{Z}_2[x]/(x^3 + x + 1)$: (a) $(x + 1)^{20}$
(b) $(x + 1)(x^2 + 1)(x^3 + 1)$ (c) $1/(x^2 + x)$.

8. Factor over $\mathbb{Z}_2$: (a) $x^7 - 1$ (b) $x^8 - 1$ (c) $x^9 - 1$.

9. Write the generator matrix and the parity check matrix for the length 6 cyclic code with the generator polynomial $g(x) = x^2 + x + 1$. How many errors does this code correct?

10. How many irreducible factors are there in the decomposition of $x^{127} - 1$ over $\mathbb{Z}_2$? What degrees do they have?