MAT 148, Winter 2016
Solutions to homework Assignment 1

Appendix: 6. (25 points) Show that any set of vectors that contains a dependent set is dependent.

Solution: Suppose that we have vectors $v_1, \ldots, v_n$ and the vectors $v_1, \ldots, v_k$ are dependent for some $k \le n$. Then there exist coefficients $\alpha_1, \ldots, \alpha_k$ such that at least one α_i does not vanish and $\alpha_1 v_1 + \ldots + \alpha_k v_k = 0$. Define $\alpha_{k+1} = \ldots = \alpha_n = 0$, then

$$\alpha_1 v_1 + \ldots + \alpha_n v_n = \alpha_1 v_1 + \ldots + \alpha_k v_k + 0 v_{k+1} + \ldots + 0 v_n = \alpha_1 v_1 + \ldots + \alpha_k v_k = 0,$$

so $v_1, \ldots, v_n$ are dependent.

7. (25 points) Show that any set of vectors that is contained in an independent set is independent.

Solution: Suppose that $v_1, \ldots, v_n$ are independent, let us prove that $v_1, \ldots, v_k$ are independent for all $k \le n$. Indeed, if they would be dependent, by the previous problem $v_1, \ldots, v_n$ would be dependent. Contradiction.

16. (25 points) Let

$$M = \begin{pmatrix} 0 & 1 & 0 & 0 & 2 \\ 2 & 3 & 1 & 4 & 0 \\ 2 & 1 & 0 & 1 & 1 \\ 5 & 0 & 0 & 3 & 4 \end{pmatrix}.$$

Find a sequence of elementary operations showing that premultiplying M by the corresponding elementary matrices results in the row-echelon form.

Solution: Let us add the identity matrix to the right of M:

$$M' = \begin{pmatrix} 0 & 1 & 0 & 0 & 2| & 1 & 0 & 0 & 0 \\ 2 & 3 & 1 & 4 & 0| & 0 & 1 & 0 & 0 \\ 2 & 1 & 0 & 1 & 1| & 0 & 0 & 1 & 0 \\ 5 & 0 & 0 & 3 & 4| & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Divide the second row by 2 and swap it with the first:

$$\begin{pmatrix} 1 & 3/2 & 1/2 & 2 & 0| & 0 & 1/2 & 0 & 0 \\ 0 & 1 & 0 & 0 & 2| & 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 1 & 1| & 0 & 0 & 1 & 0 \\ 5 & 0 & 0 & 3 & 4| & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Subtract the first row multiplied by 2 and by 5 from the third and fourth rows:

$$\begin{pmatrix} 1 & 3/2 & 1/2 & 2 & 0| & 0 & 1/2 & 0 & 0 \\ 0 & 1 & 0 & 0 & 2| & 1 & 0 & 0 & 0 \\ 0 & -2 & -1 & -3 & 1| & 0 & -1 & 1 & 0 \\ 0 & -15/2 & -5/2 & -7 & 4| & 0 & -5/2 & 0 & 1 \end{pmatrix}.$$

Add the second row multiplied by 2 and by 15/2 to the third and fourth rows:

$$\begin{pmatrix} 1 & 3/2 & 1/2 & 2 & 0 & | & 0 & 1/2 & 0 & 0 \\ 0 & 1 & 0 & 0 & 2 & | & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & -3 & 5 & | & 2 & -1 & 1 & 0 \\ 0 & 0 & -5/2 & -7 & 19 & | & 15/2 & -5/2 & 0 & 1 \end{pmatrix}.$$

Multiply the third row by (-1):

$$\begin{pmatrix} 1 & 3/2 & 1/2 & 2 & 0 & | & 0 & 1/2 & 0 & 0 \\ 0 & 1 & 0 & 0 & 2 & | & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 3 & -5 & | & -2 & 1 & -1 & 0 \\ 0 & 0 & -5/2 & -7 & 19 & | & 15/2 & -5/2 & 0 & 1 \end{pmatrix}.$$

Add the 3rd row multiplied by 5/2 to the fourth one:

$$\begin{pmatrix} 1 & 3/2 & 1/2 & 2 & 0 & | & 0 & 1/2 & 0 & 0 \\ 0 & 1 & 0 & 0 & 2 & | & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 3 & -5 & | & -2 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1/2 & 13/2 & | & 5/2 & 0 & -5/2 & 1 \end{pmatrix}.$$

Multiply the 4th row by 2:

$$\begin{pmatrix} 1 & 3/2 & 1/2 & 2 & 0 & | & 0 & 1/2 & 0 & 0 \\ 0 & 1 & 0 & 0 & 2 & | & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 3 & -5 & | & -2 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & 13 & | & 5 & 0 & -5 & 2 \end{pmatrix}.$$

Subtract the 4th row multiplied by 2 and by 3 from 1st and 3rd:

$$\left(\begin{array}{ccccc|cccc} 1 & 3/2 & 1/2 & 0 & -26 & -10 & 1/2 & 10 & -4 \\ 0 & 1 & 0 & 0 & 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -44 & -17 & 1 & 14 & -6 \\ 0 & 0 & 0 & 1 & 13 & 5 & 0 & -5 & 2 \end{array}\right).$$

Subtract the second row multiplied by 3/2 and the 3rd row multiplied by 1/2 from the first row:

$$\left(\begin{array}{ccccc|cccc} 1 & 0 & 0 & 0 & -7 & -3 & 0 & 10 & -1 \\ 0 & 1 & 0 & 0 & 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -44 & -17 & 1 & 14 & -6 \\ 0 & 0 & 0 & 1 & 13 & 5 & 0 & -5 & 2 \end{array}\right).$$

On the left we get the reduced row-echelon form of $R(M)$ and on the right the matrix A such that $AM = R(M)$.

A. (25 points) How many vectors are there in a d-dimensional space over a field with 2 elements?

Solution: Every such vector corresponds to a binary sequence $(x_1, \ldots, x_d)$, where each x_i can equal 0 or 1. Since there are 2 choices for each x_i and these choices are independent, the total number of length d binary sequences is 2^d.