MAT 148, Winter 2016
Solutions to HW 3

4. (25 points) Consider the [6,3] code with the generator matrix:

\[ G = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{pmatrix} \]

Find all the cosets, their syndromes and minimal weights in each coset.

**Solution:** First we need to apply the row reduction to \( G \) (subtract the 3rd row from the 2nd):

\[ G' = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{pmatrix} \]

The parity check matrix then equals

\[ H = \begin{pmatrix} 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \end{pmatrix} \]

We can use it to compute the syndromes of all vectors and list all the cosets:

<table>
<thead>
<tr>
<th>Leader</th>
<th>Coset</th>
<th>Syndrome</th>
<th>Min. wt</th>
</tr>
</thead>
<tbody>
<tr>
<td>000000</td>
<td>000000,100011,010111,001111,110100,101110,011001,111101,101010</td>
<td>000</td>
<td>0</td>
</tr>
<tr>
<td>100000</td>
<td>000000,000011,110111,101110,010100,001101,111001,011010</td>
<td>011</td>
<td>1</td>
</tr>
<tr>
<td>010000</td>
<td>010000,110011,000111,011110,100100,111101,001001,101010</td>
<td>111</td>
<td>1</td>
</tr>
<tr>
<td>001000</td>
<td>001000,101011,011111,000110,111100,100101,010001,110010</td>
<td>110</td>
<td>1</td>
</tr>
<tr>
<td>000100</td>
<td>000100,100111,010011,001010,110000,101001,011101,111110</td>
<td>100</td>
<td>1</td>
</tr>
<tr>
<td>000010</td>
<td>000010,100011,010101,001100,110110,101111,010111,111000</td>
<td>010</td>
<td>1</td>
</tr>
<tr>
<td>000001</td>
<td>000001,100010,010110,001111,110101,101100,011000,111011</td>
<td>001</td>
<td>1</td>
</tr>
<tr>
<td>101000</td>
<td>101000,001011,111111,100110,011100,000101,110001,010010</td>
<td>101</td>
<td>2</td>
</tr>
</tbody>
</table>

9. (25 points) Is there a [12,7,5] binary code (that is, \( n = 12, k = 7 \) and the minimal distance between codewords is 5)?
Solution: Suppose that such code exists. Since the minimal distance between codewords is \( d = 5 \), then this code corrects \( t = (5 - 1)/2 = 2 \) errors, and we can apply the Sphere-Packing Bound:

\[
\left(1 + 12 + \binom{12}{2}\right) 2^7 \leq 2^{12}, 1 + 12 + 66 = 79 \leq 2^5 = 32.
\]

Contradiction, so there is no such code.

13. (25 points) Prove that any binary [23,12,7] code is perfect. How many errors can it correct?

Solution: Since the minimal distance between codewords is \( d = 7 \), then this code corrects \( t = (7 - 1)/2 = 3 \) errors. We can apply the Sphere-Packing Bound:

\[
\left(1 + 23 + \binom{23}{2} + \binom{23}{3}\right) 2^{12} \leq 2^{23}.
\]

Remark that

\[
1+23+\binom{23}{2}+\binom{23}{3} = 1+23+\frac{23 \cdot 22}{2} + \frac{23 \cdot 22 \cdot 21}{6} = 1+23+253+1771 = 2048,
\]

and \( 2048 \cdot 2^{12} = 2^{11} \cdot 2^{12} = 2^{23} \). Since we get an equality in the Sphere-Packing Bound, the code is perfect.

B. (25 points) Decode the message \( y = 1001001 \), which was encoded using the [7,4] Hamming code with the generator matrix

\[
G = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 1 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 1 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 1 & 1 & 1 & 1
\end{pmatrix}
\]

and at most one bit was changed.

Solution: The parity check matrix has the form:

\[
H = \begin{pmatrix}
0 & 1 & 1 & 1 & 1 & 0 & 0 \\
1 & 0 & 1 & 1 & 0 & 1 & 0 \\
1 & 1 & 0 & 1 & 0 & 0 & 1
\end{pmatrix}
\]

Let us check all possible options for \( x \) (different from \( y \) in at most one bit) and compute their syndromes:
A word $x$ is a codeword if and only if its syndrome equals to 0, so the corresponding codeword is 1101001. The original message consists of the first 4 bits of $x$ and reads as 1101.