MAT 148, Winter 2016
Solutions to HW 3

4. (25 points) Consider the [6,3] code with the generator matrix:

$$G = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{pmatrix}$$

Find all the cosets, their syndromes and minimal weights in each coset.

Solution: First we need to apply the row reduction to G (subtract the 3rd row from the 2nd):

$$G' = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{pmatrix}$$

The parity check matrix then equals

$$H = \begin{pmatrix} 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \end{pmatrix}$$

We can use it to compute the syndromes of all vectors and list all the cosets:

Leader	Coset	Syndrome	Min. wt
000000	000000,100011,010111,001110,110100,101101,011001,111010	000	0
100000	100000,000011,110111,101110,010100,001101,111001,011010	011	1
010000	010000,110011,000111,011110,100100,111101,001001,101010	111	1
001000	001000,101011,011111,000110,111100,100101,010001,110010	110	1
000100	000100,100111,010011,001010,110000,101001,011101,111110	100	1
000010	000010,100001,010101,001100,110110,101111,011011,111000	010	1
000001	000001,100010,010110,001111,110101,101100,011000,111011	001	1
101000	101000,001011,111111,100110,011100,000101,110001,010010	101	2

9. (25 points) Is there a [12,7,5] binary code (that is, $n = 12, k = 7$ and the minimal distance between codewords is 5)?

Solution: Suppose that such code exists. Since the minimal distance between codewords is $d = 5$, then this code corrects $t = (5-1)/2 = 2$ errors, and we can apply the Sphere-Packing Bound:

$$\left(1 + 12 + \binom{12}{2}\right) 2^7 \leq 2^{12}, 1 + 12 + 66 = 79 \leq 2^5 = 32.$$

Contradiction, so there is no such code.

13. (25 points) Prove that any binary [23,12,7] code is perfect. How many errors can it correct?

Solution: Since the minimal distance between codewords is $d = 7$, then this code corrects $t = (7-1)/2 = 3$ errors. We can apply the Sphere-Packing Bound:

$$\left(1 + 23 + \binom{23}{2} + \binom{23}{3}\right) 2^{12} \leq 2^{23}.$$

Remark that

$$1+23+\binom{23}{2}+\binom{23}{3} = 1+23+\frac{23 \cdot 22}{2}+\frac{23 \cdot 22 \cdot 21}{6} = 1+23+253+1771 = 2048,$$

and $2048 \cdot 2^{12} = 2^{11} \cdot 2^{12} = 2^{23}$. Since we get an equality in the Sphere-Packing Bound, the code is perfect.

B. (25 points) Decode the message $y = 1001001$, which was encoded using the [7,4] Hamming code with the generator matrix

$$G = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{pmatrix}$$

and at most one bit was changed.

Solution: The parity check matrix has the form:

$$H = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 1 \end{pmatrix}$$

Let us check all possible options for x (different from y in at most one bit) and compute their syndromes:

x	1001001	0001001	1101001	1011001	1000001	1001101	1001011	1001000
$syn(x)$	101	110	000	011	010	001	111	100

A word x is a codeword if and only if it syndrome equals to 0, so the corresponding codeword is 1101001. The original message consists of the first 4 bits of x and reads as 1101.