# MAT 150A, Fall 2015
# Practice problems for the final exam

1. Let $f : S_n \to G$ be any homomorphism (to some group $G$) such that $f(1\ 2) = e$. Prove that $f(x) = e$ for all $x$.
2. Are the following subsets of $D_n$ subgroups? Normal subgroups?
a) All reflections in $D_n$
b) All rotations in $D_n$
c) $\{1, s\}$ where $s$ is some reflection
3. Consider the set

$$G = \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} : a \neq 0 \right\}$$

and a function $f : G \to \mathbb{R}^*$,

$$f \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} = a.$$

a) Prove that $G$ is a subgroup of $GL_2$
b) Prove that $f$ is a homomorphism.
c) Find the kernel and image of $f$.
4. Consider the permutation

$$f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 5 & 6 & 1 & 7 & 3 & 2 & 4 \end{pmatrix}$$

a) Decompose $f$ into non-intersecting cycles
b) Find the order of $f$
c) Find the sign of $f$
d) Compute $f^{-1}$
5. Find all possible orders of elements in $D_6$.
6. a) Prove that every rotation of the plane is a composition of two reflections. What is the angle between the reflecting lines?
b) Prove that every rotation is conjugate to its inverse in $D_n$.
7. The *trace* of a $2 \times 2$ matrix is defined as

$$\operatorname{tr} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = a + d.$$

a) Prove that $\operatorname{tr}(AB) = \operatorname{tr}(BA)$ for all $A$ and $B$

b) Prove that $\operatorname{tr}(A^{-1}XA) = \operatorname{tr}(X)$, so the conjugate matrices have the same trace.

8. Let $A$ be the counterclockwise rotation by $90°$, let $B$ be the reflection in the line $\{x = y\}$. Present the transformation $A, B, AB, BA$ by matrices, describe $AB$ and $BA$ geometrically.

9. Are there two non-isomorphic groups with (a) 6 elements (b) 7 elements (c) 8 elements?

10. Is it possible to construct a surjective homomorphism from a group with 6 elements to a group with (a) 7 elements (b) 5 elements (c) 3 elements? If yes, construct such a homomorphism. If no, explain why this is not possible.

11. Is it possible to construct an injective homomorphism from a group with 6 elements to a group with (a) 3 elements (b) 9 elements (c) 12 elements? If yes, construct such a homomorphism. If no, explain why this is not possible.

12. Are the following matrices orthogonal? Do they preserve orientation?

$$\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & -\frac{2}{\sqrt{6}} \end{pmatrix}.$$

13. Prove that for every $n$ there is a group with $n$ elements.

14. Solve the system of equations

$$\begin{cases} x = 3 \mod 5 \\ x = 4 \mod 6. \end{cases}$$

15. Compute $3^{100} \mod 7$.

16. A *triangulation* of an $n$-gon is a collection of $(n-3)$ non-intersecting diagonals. For $n = 4, 5, 6$ :

a) Find the total number of triangulations for a regular $n$-gon

b) Describe the orbits and stabilizers of the action of $D_n$ on the set of triangulations.