MAT 150A, Fall 2015
Practice problems for the final exam

1. Let $f : S_n \to G$ be any homomorphism (to some group G) such that $f(1\ 2) = e$. Prove that $f(x) = e$ for all x.

Solution: The kernel of f is a normal subgroup in S_n containing $(1\ 2)$. Since it is normal, it also contains all transpositions. Since it is closed under multiplication and every permutation is a product of transpositions, the kernel coincides with the whole S_n, so $f(x) = e$ for all x.

2. Are the following subsets of D_n subgroups? Normal subgroups?
a) All reflections in D_n
b) All rotations in D_n
c) $\{1, s\}$ where s is some reflection

Solution: (a) No: it does not contain the identity! (b) Yes, it is a normal subgroup. It contains the identity, the product of two rotations is a rotation and the inverse of a rotation is a rotation, so it is a subgroup. Also, $\det(g^{-1}xg) = \det(x)$, so any matrix conjugate to a rotation must have determinant 1 and hence is a rotation, so it is normal. (c) It is clearly a cyclic subgroup generated by s, but it is not normal: if s_1 is some other reflection then one can check (see also problem 6) that $(s_1)^{-1}ss_1 \neq s$.

3. Consider the set

$$G = \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} : a \neq 0 \right\}$$

and a function $f : G \to \mathbb{R}^*$,

$$f \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} = a.$$

a) Prove that G is a subgroup of GL_2
b) Prove that f is a homomorphism.
c) Find the kernel and image of f.

Solution: a) We have to check 3 defining properties a subgroup:

- Identity is in G: take $a = 1, b = 0$

- Closed under multiplication:

$$\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a_1 & b_1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} aa_1 & ab_1 + b \\ 0 & 1 \end{pmatrix}$$

- Closed under taking inverses: we need $aa_1 = 1, ab_1 + b = 0$, so

$$a_1 = 1/a,\ b_1 = -b/a.$$

b) From (a) we see that $f(AB) = aa_1 = f(A)f(B)$.
c) Since a can be arbitrary, $Im(f) = \mathbb{R}^*$. Now

$$Ker(f) = \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} : b \text{ arbitrary} \right\}.$$

4. Consider the permutation

$$f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 5 & 6 & 1 & 7 & 3 & 2 & 4 \end{pmatrix}$$

a) Decompose f into non-intersecting cycles
b) Find the order of f
c) Find the sign of f
d) Compute f^{-1}

Solution: $f = (1\ 5\ 3)(2\ 6)(4\ 7)$, it has order $lcm(3, 2, 2) = 6$ and sign $(-1)^{3-1}(-1)(-1) = 1$, $f^{-1} = (1\ 3\ 5)(2\ 6)(4\ 7)$.

5. Find all possible orders of elements in D_6.

Solution: The group contains reflections and rotations by multiples of $360°/6 = 60°$. Every reflection has order 2, identity has order 1, rotations by $60°$ and $300°$ have order 6, rotations by $120°$ and $240°$ have order 3 and rotation by $180°$ has order 2.

6. a) Prove that every rotation of the plane is a composition of two reflections. What is the angle between the reflecting lines?
b) Prove that every rotation is conjugate to its inverse in D_n.

Solution: One can check geometrically that if s_1 and s_2 are two reflections, then $s_1 s_2$ and $s_2 s_1$ are rotations by angles 2α and -2α, where α is the angle between reflecting lines. (a) Given a rotation R by angle α, we can pick two lines with angle $\alpha/2$ and by the above $R = s_1 s_2$, $R^{-1} = s_2 s_1$. (b) Now

$$R^{-1} = s_2 s_1 = s_2(s_1 s_2)s_2^{-1} = s_2 R s_2^{-1},$$

so R and R^{-1} are conjugate by s_2.

7. The *trace* of a 2×2 matrix is defined as

$$\mathrm{tr}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = a + d.$$

a) Prove that $\mathrm{tr}(AB) = \mathrm{tr}(BA)$ for all A and B
b) Prove that $\mathrm{tr}(A^{-1}XA) = \mathrm{tr}(X)$, so the conjugate matrices have the same trace.

Solution: a) We have

$$AB = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} = \begin{pmatrix} aa_1 + bc_1 & ab_1 + bd_1 \\ ca_1 + dc_1 & cb_1 + dd_1 \end{pmatrix},$$

so $\mathrm{tr}(AB) = aa_1 + bc_1 + cb_1 + dd_1$. Clearly, this expression will not change if we swap A and B.

b) By part (a), $\mathrm{tr}(A^{-1} \cdot XA) = \mathrm{tr}(XA \cdot A^{-1}) = \mathrm{tr}(X)$.

8. Let A be the counterclockwise rotation by $90°$, let B be the reflection in the line $\{x = y\}$. Present the transformation A, B, AB, BA by matrices, describe AB and BA geometrically.

Solution:

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, AB = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, BA = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

so AB and BA are reflections in the y-axis and in the x-axis respectively.

9. Are there two non-isomorphic groups with (a) 6 elements (b) 7 elements (c) 8 elements?

Solution: (a) Yes, for example D_3 and $\mathbb{Z}_6$. The orders of elements in D_3 are 1,2,3, while $\mathbb{Z}_6$ has an element of order 6, so they are not isomorphic. (b) No: by Lagrange theorem the order of every element x divides 7, so it should be 1 (and $x = e$) or 7. If the order of x equals 7, then this is just the cyclic group generated by x. So every two groups with 7 elements are cyclic and hence isomorphic. (c) Yes, for example D_4 and $\mathbb{Z}_8$. The orders of elements in D_4 are 1,2,4, while $\mathbb{Z}_8$ has an element of order 8, so they are not isomorphic.

10. Is it possible to construct a surjective homomorphism from a group with 6 elements to a group with (a) 7 elements (b) 5 elements (c) 3 elements? If yes, construct such a homomorphism. If no, explain why this is not possible.

Solution: (a) No, since $7 > 6$. (b) No, since by counting formula the size of the image should divide 6. (c) Yes, for example

$$f : \mathbb{Z}_6 \to \mathbb{Z}_3,\ f(k) = k \mod 3.$$

11. Is it possible to construct an injective homomorphism from a group with 6 elements to a group with (a) 3 elements (b) 9 elements (c) 12 elements? If yes, construct such a homomorphism. If no, explain why this is not possible.

Solution: The image of an injective homomorphism should be a subgroup with 6 elements. (a) No, since $3 < 6$. (b) No, since by Lagrange Theorem the size of the image divides 9 (c) Yes, for example

$$f : \mathbb{Z}_6 \to \mathbb{Z}_12,\ f(k) = 2k \mod 12.$$

12. Are the following matrices orthogonal? Do they preserve orientation?

$$\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & -\frac{2}{\sqrt{6}} \end{pmatrix}.$$

Solution: A matrix is orthogonal if $A^t A = I$, and preserves orientation if $\det(A) > 0$, so: (a) Not orthogonal, preserves. (b) Orthogonal, reverses. (c) Orthogonal, preserves.

13. Prove that for every n there is a group with n elements.

Solution: Indeed, consider cyclic group $\mathbb{Z}_n$.

14. Solve the system of equations

$$\begin{cases} x = 3 \mod 5 \\ x = 4 \mod 6. \end{cases}$$

Solution: By Chinese Remainder Theorem, the solution is unique modulo 30. Since $x = 4 \mod 6$, we get

$$x = 4, 10, 16, 22, 28 \mod 30$$

Among these values, only $x = 28$ is equal to 3 modulo 5.

15. Compute $3^{100} \mod 7$.

Solution: We have

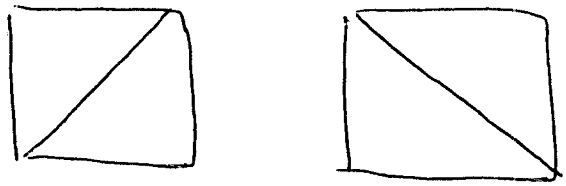
$$3^1 = 3,\ 3^2 = 2,\ 3^3 = 2{\cdot}3 = 6,\ 3^4 = 6{\cdot}3 = 4,\ 3^5 = 4{\cdot}3 = 5,\ 3^6 = 5{\cdot}3 = 1 \mod 7$$

Therefore

$$3^{100} = (3^6)^{16} \cdot 3^4 = 1^{16} \cdot 4 = 4 \mod 7.$$

16. A *triangulation* of an n-gon is a collection of $(n-3)$ non-intersecting diagonals. For $n = 4, 5, 6$:
a) Find the total number of triangulations for a regular n-gon
b) Describe the orbits and stabilizers of the action of D_n on the set of triangulations.

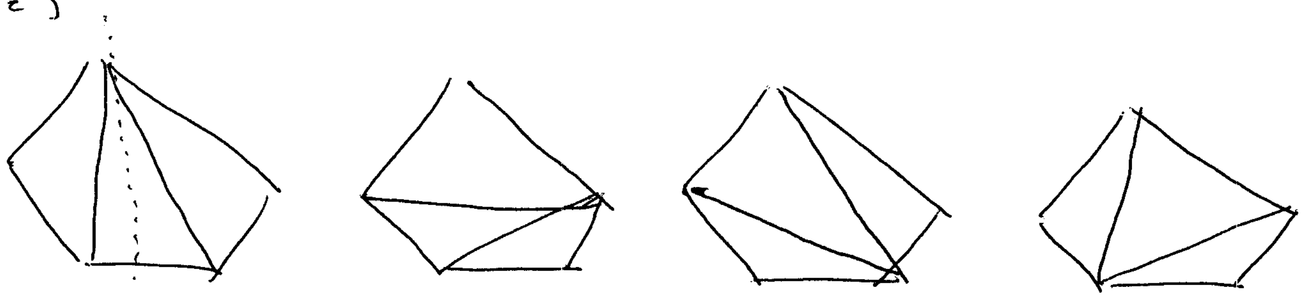
16 a) $n=4$

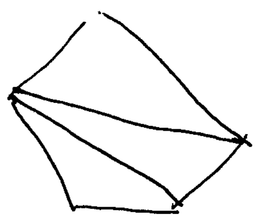


One orbit with 2 elements,
Stabilizer $= \{e, \text{reflections in diagonals}, \text{rotation by } 180°\}$

$2 \cdot 4 = 8 = |D_4|$

b) $n=5$

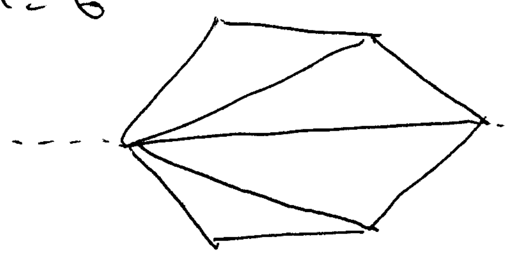




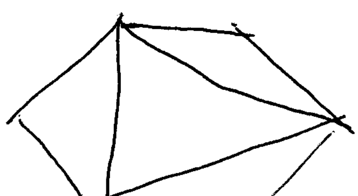
5 triangulations in a single orbit
Stabilizer $= \{e, \text{reflection}\}$

$5 \cdot 2 = 10 = |D_5|$

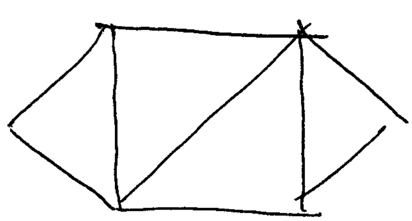
c) $n=6$



$\times 6$: $|\text{Orbit}| = 6$
Stabilizer $= \{e, \text{reflection}\}$
$6 \cdot 2 = 12 = |D_6|$



$\times 2$: $|\text{Orbit}| = 2$
Stabilizer $= D_3$
$2 \cdot 6 = 12 = |D_6|$



$\times 6$: $|\text{Orbit}| = 2$
Stabilizer $= \{e, \text{rotation by } 180°\}$
$2 \cdot 6 = 12 = |D_6|$

In total, 14 triangulations of a 6-gon