2.4. (25 points) Determine if the following subset $H$ is a subgroup of $G$:

b) (8 points) $G = \mathbb{R}^*$ and $H = \{1,-1\}$.

**Solution:** This is a subgroup, in fact, $H$ is a cyclic subgroup of $\mathbb{R}^*$ generated by $(-1)$.

c) (8 points) $G = \mathbb{Z}^+$ and $H$ is the set of positive integers.

**Solution:** This is not a subgroup since 0 is not positive and every subgroup should contain 0. Also, 2 is in $H$, but $(-2)$ is not in $H$, so $H$ is not closed under taking (additive) inverses.

d) (9 points) $G = \mathbb{R}^*$ and $H$ is the set of positive reals.

**Solution:** This is a subgroup: indeed, $1 > 0$; if $x, y > 0$ then $xy > 0$, so $H$ is closed under multiplication; if $x > 0$ then $1/x > 0$, so $H$ is closed under taking inverses.

4.7. (25 points) Let $x$ and $y$ be elements of a group $G$. Assume that each of the elements $x$, $y$ and $xy$ has order 2. Prove that the set $H = \{1, x, y, xy\}$ is a subgroup of $G$.

**Solution:** Let us prove that $xy = yx$. Indeed, $(xy)^2 = 1$, so $xyxy = 1$. If we multiply this by $x$ on the left and use the equation $x^2 = 1$, we get $yxy = x$. If we multiply this by $y$ on the left and use the equation $y^2 = 1$, we get $xy = yx$.

Now since $x, y, xy$ have order 2, they are all inverses to themselves, so $H$ is closed under taking inverses. To check that it is closed under multiplication, it remains to notice that (since $xy = yx$):

$$xy \cdot x = x \cdot xy = y,$$

$$xy \cdot y = y \cdot xy = x.$$

4.9. (25 points) How many elements of order 2 does the symmetric group $S_4$ contain?

**Solution:** A permutation has order 2 if and only if all non-intersecting cycles in it have length 2, that is, are transpositions. In $S_4$, such a permutation can a transposition:

$$(1 \, 2), \ (1 \, 3), \ (1 \, 4), \ (2 \, 3), \ (2 \, 4), \ (3 \, 4),$$
or a product of 2 transpositions:

\((1 \, 2)(3 \, 4), \, (1 \, 3)(2 \, 4), \, (1 \, 4)(2 \, 3)\).

In total, we get 9 permutations of order 2 in \(S_4\).

C. (25 points) Find the order of the permutation

\[ f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 3 & 10 & 7 & 6 & 2 & 5 & 9 & 4 & 1 & 8 \end{pmatrix}. \]

**Solution:** We can decompose this permutation into non-intersecting cycles:

\[ f = (1 \, 3 \, 7 \, 9)(2 \, 10 \, 8 \, 4 \, 6 \, 5). \]

Since we get two cycles of orders 4 and 6, the order of \(f\) equals \(lcm(4, 6) = 12\).