6.2. (25 points) Describe all homomorphisms \( \varphi : \mathbb{Z} \to \mathbb{Z} \). Determine which are injective, which are surjective and which are isomorphisms.

**Solution:** If \( \varphi \) is an homomorphism then \( \varphi(0) = 0 \), and for all \( x, y \) \( \varphi(x + y) = \varphi(x) + \varphi(y) \). Suppose that \( \varphi(1) = n \), then \( \varphi(2) = \varphi(1 + 1) = \varphi(1) + \varphi(1) = 2n \). Similarly (one can prove this by induction), for all \( k > 0 \) \( \varphi(k) = kn \). Now \( \varphi(-k) + \varphi(k) = \varphi(0) = 0 \), so \( \varphi(-k) = -\varphi(k) = -kn \). Therefore for all \( x \) we have \( \varphi(x) = nx \). Indeed, such function is a homomorphism.

It is injective if and only if \( \ker \varphi = \{0\} \), that is, if and only if \( n \neq 0 \). Since all elements in the image of \( \varphi \) are divisible by \( n \), it is surjective for \( n = \pm 1 \). As a result, \( \varphi \) is an isomorphism if and only if \( n = \pm 1 \), so \( \varphi(x) = x \) or \( \varphi(x) = -x \).

6.3. (25 points) Show that the functions \( f = 1/x, g = (x - 1)/x \) generate the group of functions, the law of composition being composition of functions, that is isomorphic to the symmetric group \( S_3 \).

**Solution:** Let us compute various compositions of \( f \) and \( g \):

\[
\begin{align*}
    f(f(x)) &= x, \quad f(g(x)) = \frac{x}{x - 1}, \quad g(f(x)) = \frac{1}{x - 1} = 1 - x, \\
    g(g(x)) &= \frac{(x - 1)/x}{x} = \frac{x - 1 - x}{x - 1} = 1 - \frac{1}{x} = \frac{1}{1 - x}.
\end{align*}
\]

Together with \( f \) and \( g \), we get six different different functions, and we can describe all compositions between them: in the table below the cell in row \( A \) and column \( B \) contains \( A(B(x)) \).

\[
\begin{array}{cccccccc}
  e = x & f = 1/x & g = (x - 1)/x & fg = x/(x - 1) & gf = 1 - x & g^2 = 1/(1 - x) \\
  \hline
  x & x & 1/x & (x - 1)/x & x/(x - 1) & 1 - x & 1/(1 - x) \\
  1/x & 1/x & x & x/(x - 1) & (x - 1)/x & 1/(1 - x) & 1 - x \\
  (x - 1)/x & (x - 1)/x & 1 - x & 1/(1 - x) & x/(x - 1) & 1 - x & x/(x - 1) \\
  x/(x - 1) & x/(x - 1) & 1 - x & 1/(1 - x) & x & 1 - x & (x - 1)/x \\
  1 - x & 1 - x & (x - 1)/x & 1/x & 1/(1 - x) & x & x/(x - 1) \\
  1/(1 - x) & 1/(1 - x) & x & 1 - x & 1/x & (x - 1)/x & \phantom{1/(1 - x)}
\end{array}
\]

From the table, we see that \( g(g(g(x))) = x \), so \( g \) has order 3, while \( f \) clearly has order 2. Consider the function \( M : G \to S_3 \) which is uniquely defined by the equations:

\[
M(f) = (1 \ 2), \quad M(g) = (1 \ 2 \ 3).
\]

We can fill in the analogous multiplication table for \( S_3 \):

\[
\begin{array}{cccccccc}
  e & (1) & (2) & (1 \ 2 \ 3) & (2 \ 3) & (1 \ 3) & (1 \ 3 \ 2) \\
  \hline
  (1) & e & (1 \ 2) & (1 \ 2 \ 3) & (2 \ 3) & (1 \ 3) & (1 \ 3 \ 2) \\
  (2) & (1 \ 2) & e & (1 \ 2 \ 3) & (1 \ 3 \ 2) & (1 \ 2) & (1 \ 3) \\
  (1 \ 2 \ 3) & (1 \ 3 \ 2) & (1 \ 2 \ 3) & e & (1 \ 2) & (1 \ 3 \ 2) & (1 \ 2 \ 3) \\
  (1 \ 3) & (1 \ 3) & (1 \ 2 \ 3) & (1 \ 2) & (1 \ 3 \ 2) & e & (1 \ 2 \ 3) \\
  (1 \ 3 \ 2) & (1 \ 3 \ 2) & (2 \ 3) & (1 \ 3) & (1 \ 2) & (1 \ 2 \ 3) & e \\
\end{array}
\]
We see that $M$ transforms the first table to the second one, so it identifies products in two groups, and hence $M$ is a homomorphism. Since it is clearly bijective, it is an isomorphism.

8.6. (25 points) Let $\varphi : G \to G'$ be a group homomorphism. Suppose that $|G| = 18, |G'| = 15$ and that $\varphi$ is not the trivial homomorphism. What is the order of the kernel?

**Solution:** By Counting Formula, $|\text{Ker}(\varphi)| \cdot |\text{Im}(\varphi)| = 18$, so $|\text{Im}(\varphi)|$ divides 18. Since $\text{Im}(\varphi)$ is a subgroup of $G'$, by Lagrange Theorem $|\text{Im}(\varphi)|$ also divides 15. Therefore $|\text{Im}(\varphi)|$ can be equal to 1 or 3. If $|\text{Im}(\varphi)| = 1$, then $\varphi$ is trivial. We conclude that $|\text{Im}(\varphi)| = 3$ and $|\text{Ker}(\varphi)| = 6$.

8.9. (25 points) Let $G$ be a finite group. Under which circumstances is the map $\varphi(x) = x^2$ an automorphism of $G$?

**Solution:** If $\varphi$ is a homomorphism, then for all $x, y$ $(xy)^2 = xyxy = x^2y^2$. If we multiply this equation by $x^{-1}$ on the left and by $y^{-1}$ on the right, we get $yx = xy$. Therefore $\varphi$ is a homomorphism if and only if $G$ is commutative.

To check if it is an isomorphism, it is sufficient to check if it has trivial kernel. This kernel consists of identity and all elements of order 2. Therefore $\varphi$ is an isomorphism if and only if $G$ is commutative and it does not contain elements of order 2.