2.3. Find all real $2 \times 2$ matrices that carry the line $y = x$ to the line $y = 3x$.

Solution: Let

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

then for $y = x$ we get

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ x \end{pmatrix} = \begin{pmatrix} ax + bx \\ cx + dx \end{pmatrix},$$

so $cx + dx = 3(ax + bx)$ for all $x$, and $c + d = 3(a + b)$.

D. Suppose that $M$ is an $n \times n$ matrix of finite order. Find all possible values for $\det(M)$.

Solution: If $M^k = I$ then $\det(M^k) = \det(M)^k = 1$, and $\det(M) = \pm 1$. Therefore if $M$ has finite order then $\det(M) = \pm 1$.

E. Consider the set

$$B = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} : a, c \neq 0 \right\}$$

1) Prove that $B$ is a subgroup of $GL_2$

2) Find all elements of $B$ of finite order.

Solution: 1) We have

$$\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \begin{pmatrix} a' & b' \\ 0 & c' \end{pmatrix} = \begin{pmatrix} aa' & ab' + bc' \\ 0 & cc' \end{pmatrix},$$

so $B$ is closed under multiplication. Clearly, $I \in B$ since one can set $a = c = 1, b = 0$. To find the inverse matrix, we need to solve the equations

$$aa' = 1, \quad cc' = 1, \quad ab' + bc' = 0 \Rightarrow a' = 1/a, \quad c = 1/c, \quad b' = -bc'/a = -b/ac.$$

This is well-defined provided $a, c \neq 0$.

2) Remark that

$$A = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \Rightarrow A^k = \begin{pmatrix} a^k & \ast \\ 0 & c^k \end{pmatrix}.$$

Therefore if $A$ has order $k$ then $a^k = c^k = 1$, so $a = \pm 1, c = \pm 1$. We need to consider several cases:

$$A = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \Rightarrow A^2 = \begin{pmatrix} 1 & 2b \\ 0 & 1 \end{pmatrix}, \text{ and } A^k = \begin{pmatrix} 1 & kb \\ 0 & 1 \end{pmatrix}.$$

Therefore $\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$ has finite order if and only if $b = 0$. Similarly, $\begin{pmatrix} -1 & b \\ 0 & -1 \end{pmatrix}$ has finite order if and only if $b = 0$. Furthermore,

$$A = \begin{pmatrix} 1 & b \\ 0 & -1 \end{pmatrix} \Rightarrow A^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

so $\begin{pmatrix} 1 & b \\ 0 & -1 \end{pmatrix}$ has order 2 for all $b$, and similarly $\begin{pmatrix} -1 & b \\ 0 & 1 \end{pmatrix}$ has order 2 for all $b$.

F. Find all possible $2 \times 2$ orthogonal matrices of finite order.
Solution: Every orthogonal $2 \times 2$ matrix is either a rotation or a reflection. Any reflection has order 2. If $A$ is a rotation by angle $\phi$, then $A^k$ is a rotation by $k\phi$, so $A^k = I$ if and only if

$$k\phi = 2\pi n, \quad \phi = \frac{2\pi n}{k}$$

for some $n$. Therefore all $2 \times 2$ orthogonal transformations of finite order are either reflections or rotations by rational multiples of $\pi$. 