MAT 150A, Fall 2018
Practice problems for the final exam

1. Let $f : S_n \to G$ be any homomorphism (to some group G) such that $f(1\ 2) = e$. Prove that $f(x) = e$ for all x.
2. Are the following subsets of D_n subgroups? Normal subgroups?
a) All reflections in D_n
b) All rotations in D_n
c) $\{1, s\}$ where s is some reflection
3. Consider the set

$$G = \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} : a \neq 0 \right\}$$

and a function $f : G \to \mathbb{R}^*$,

$$f \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} = a.$$

a) Prove that G is a subgroup of GL_2
b) Prove that f is a homomorphism.
c) Find the kernel and image of f.
4. Consider the permutation

$$f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 5 & 6 & 1 & 7 & 3 & 2 & 4 \end{pmatrix}$$

a) Decompose f into non-intersecting cycles
b) Find the order of f
c) Find the sign of f
d) Compute f^{-1}
5. Find all possible orders of elements in D_6.
6. a) Prove that every rotation of the plane is a composition of two reflections. What is the angle between the reflecting lines?
b) Prove that every rotation is conjugate to its inverse in D_n.
7. The *trace* of a 2×2 matrix is defined as

$$\operatorname{tr} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = a + d.$$

a) Prove that $\mathrm{tr}(AB) = \mathrm{tr}(BA)$ for all A and B
b) Prove that $\mathrm{tr}(A^{-1}XA) = \mathrm{tr}(X)$, so the conjugate matrices have the same trace.
8. Prove that the equation $x^2 + 1 = 4y$ has no integer solutions.
9. Are there two non-isomorphic groups with (a) 6 elements (b) 7 elements (c) 8 elements?
10. (a) Prove that any homomorphism from $\mathbb{Z}_{11}$ to S_{10} is trivial.
(b) Find a nontrivial homomorphism from $\mathbb{Z}_{11}$ to S_{11}.
11. How many conjugacy classes are there in S_5?
12. Are the following matrices orthogonal? Do they preserve orientation?

$$\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & -\frac{2}{\sqrt{6}} \end{pmatrix}.$$

13. Prove that for every n there is a group with n elements.
14. Solve the system of equations

$$\begin{cases} x = 1 \mod 8 \\ x = 3 \mod 6. \end{cases}$$

Is the solution unique?
15. Compute $3^{100} \mod 7$.
16. A *triangulation* of an n-gon is a collection of $(n-3)$ non-intersecting diagonals. For $n = 4, 5, 6$:
a) Find the total number of triangulations for a regular n-gon
b) Describe the orbits and stabilizers of the action of D_n on the set of triangulations.