MAT 150A, Fall 2018
Practice problems for the final exam

1. Let $f : S_n \to G$ be any homomorphism (to some group G) such that $f(1\ 2) = e$. Prove that $f(x) = e$ for all x.

Solution: The kernel of f is a normal subgroup in S_n containing $(1\ 2)$. Since it is normal, it also contains all transpositions. Since it is closed under multiplication and every permutation is a product of transpositions, the kernel coincides with the whole S_n, so $f(x) = e$ for all x.

2. Are the following subsets of D_n subgroups? Normal subgroups?
a) All reflections in D_n
b) All rotations in D_n
c) $\{1, s\}$ where s is some reflection

Solution: (a) No: it does not contain the identity! (b) Yes, it is a normal subgroup. It contains the identity, the product of two rotations is a rotation and the inverse of a rotation is a rotation, so it is a subgroup. Also, $\det(g^{-1}xg) = \det(x)$, so any matrix conjugate to a rotation must have determinant 1 and hence is a rotation, so it is normal. (c) It is clearly a cyclic subgroup generated by s, but it is not normal: if s_1 is some other reflection then one can check (see also problem 6) that $(s_1)^{-1}ss_1 \neq s$.

3. Consider the set

$$G = \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} : a \neq 0 \right\}$$

and a function $f : G \to \mathbb{R}^*$,

$$f \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} = a.$$

a) Prove that G is a subgroup of GL_2
b) Prove that f is a homomorphism.
c) Find the kernel and image of f.

Solution: a) We have to check 3 defining properties a subgroup:

- Identity is in G: take $a = 1, b = 0$

- Closed under multiplication:

$$\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a_1 & b_1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} aa_1 & ab_1 + b \\ 0 & 1 \end{pmatrix}$$

- Closed under taking inverses: we need $aa_1 = 1, ab_1 + b = 0$, so

$$a_1 = 1/a, \ b_1 = -b/a.$$

b) From (a) we see that $f(AB) = aa_1 = f(A)f(B)$.
c) Since a can be arbitrary, $Im(f) = \mathbb{R}^*$. Now

$$Ker(f) = \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} : b \text{ arbitrary} \right\}.$$

4. Consider the permutation

$$f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 5 & 6 & 1 & 7 & 3 & 2 & 4 \end{pmatrix}$$

a) Decompose f into non-intersecting cycles
b) Find the order of f
c) Find the sign of f
d) Compute f^{-1}

Solution: $f = (1\ 5\ 3)(2\ 6)(4\ 7)$, it has order $lcm(3,2,2) = 6$ and sign $(-1)^{3-1}(-1)(-1) = 1$, $f^{-1} = (1\ 3\ 5)(2\ 6)(4\ 7)$.

5. Find all possible orders of elements in D_6.

Solution: The group contains reflections and rotations by multiples of $360°/6 = 60°$. Every reflection has order 2, identity has order 1, rotations by $60°$ and $300°$ have order 6, rotations by $120°$ and $240°$ have order 3 and rotation by $180°$ has order 2.

6. a) Prove that every rotation of the plane is a composition of two reflections. What is the angle between the reflecting lines?
b) Prove that every rotation is conjugate to its inverse in D_n.

Solution: One can check geometrically that if s_1 and s_2 are two reflections, then $s_1 s_2$ and $s_2 s_1$ are rotations by angles 2α and -2α, where α is the angle between reflecting lines. (a) Given a rotation R by angle α, we can pick two lines with angle $\alpha/2$ and by the above $R = s_1 s_2$, $R^{-1} = s_2 s_1$. (b) Now

$$R^{-1} = s_2 s_1 = s_2(s_1 s_2)s_2^{-1} = s_2 R s_2^{-1},$$

so R and R^{-1} are conjugate by s_2.

7. The *trace* of a 2×2 matrix is defined as

$$\operatorname{tr}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = a + d.$$

a) Prove that $\operatorname{tr}(AB) = \operatorname{tr}(BA)$ for all A and B
b) Prove that $\operatorname{tr}(A^{-1}XA) = \operatorname{tr}(X)$, so the conjugate matrices have the same trace.

Solution: a) We have

$$AB = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} = \begin{pmatrix} aa_1 + bc_1 & ab_1 + bd_1 \\ ca_1 + dc_1 & cb_1 + dd_1 \end{pmatrix},$$

so $\operatorname{tr}(AB) = aa_1 + bc_1 + cb_1 + dd_1$. Clearly, this expression will not change if we swap A and B.

b) By part (a), $\operatorname{tr}(A^{-1} \cdot XA) = \operatorname{tr}(XA \cdot A^{-1}) = \operatorname{tr}(X)$.

8. Prove that the equation $x^2 + 1 = 4y$ has no integer solutions.

Solution: Let us consider all possible remainders of x modulo 4. If $x = 0 \mod 4$ then $x^2 + 1 = 1 \mod 4$; if $x = 1 \mod 4$ then $x^2 + 1 = 2 \mod 4$;if $x = 2 \mod 4$ then $x^2 + 1 = 1 \mod 4$;if $x = 3 \mod 4$ then $x^2 + 1 = 2 \mod 4$. Therefore $x^2 + 1$ is never divisible by 4.

9. Are there two non-isomorphic groups with (a) 6 elements (b) 7 elements (c) 8 elements?

Solution: (a) Yes, for example D_3 and $\mathbb{Z}_6$. The orders of elements in D_3 are 1,2,3, while $\mathbb{Z}_6$ has an element of order 6, so they are not isomorphic. (b) No: by Lagrange theorem the order of every element x divides 7, so it should be 1 (and $x = e$) or 7. If the order of x equals 7, then this is just the cyclic group generated by x. So every two groups with 7 elements are cyclic and hence isomorphic. (c) Yes, for example D_4 and $\mathbb{Z}_8$. The orders of elements in D_4 are 1,2,4, while $\mathbb{Z}_8$ has an element of order 8, so they are not isomorphic.

10. (a) Prove that any homomorphism from $\mathbb{Z}_{11}$ to S_{10} is trivial.
(b) Find a nontrivial homomorphism from $\mathbb{Z}_{11}$ to S_{11}.

Solution: (a) Suppose that $f : \mathbb{Z}_{11} \to S_{10}$ is a homomorphism. Then by Counting Formula $|Im(f)|$ divides $|Z_{11}| = 11$. Since 11 is prime, the image of f has either 1 or 11 elements.

On the other hand, by Lagrange Theorem $|Im(f)|$ divides $|S_{10}| = 10!$. Since 11 does not divide 10!, the image must have 1 element, so f is trivial.

(b) We can define $f(k) = (1\ 2\ 3\ 4\ 5\ 6\ 7\ 8\ 9\ 10\ 11)^k$ for all integer k. Then $f(k+l) = f(k)f(l)$ and $f(11) = e$, so f defines a homomorphism from $\mathbb{Z}_{11}$ to S_{11}.

11. How many conjugacy classes are there in S_5?

Solution: The conjugacy classes in S_n correspond to cycle types. There are 7 possible cycle types (listed by length of their cycles): $e, 2, 3, 4, 5, 2+2, 2+3$.

12. Are the following matrices orthogonal? Do they preserve orientation?

$$\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & -\frac{2}{\sqrt{6}} \end{pmatrix}.$$

Solution: A matrix is orthogonal if $A^t A = I$, and preserves orientation if $\det(A) > 0$, so: (a) Not orthogonal, preserves. (b) Orthogonal, reverses. (c) Orthogonal, preserves.

13. Prove that for every n there is a group with n elements.

Solution: Indeed, consider cyclic group $\mathbb{Z}_n$.

14. Solve the system of equations

$$\begin{cases} x = 1 \mod 8 \\ x = 3 \mod 6. \end{cases}$$

Solution: We cannot apply Chinese Remainder Theorem since 6 and 8 are not coprime. Nevertheless, the remainders of $x \mod 6$ and $\mod 8$ do not change if we add $LCM(6,8) = 24$ to x, so we can consider $x \mod 24$. Since $x = 1 \mod 8$, we get

$$x = 1, 9, 17 \mod 24$$

Among these values, only $x = 9$ is equal to 3 modulo 6.

Answer: $x = 9 \mod 24$.

15. Compute $3^{100} \mod 7$.

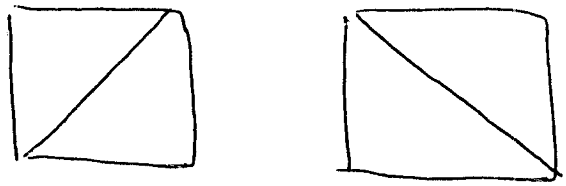
Solution: We have

$$3^1 = 3,\ 3^2 = 2,\ 3^3 = 2{\cdot}3 = 6,\ 3^4 = 6{\cdot}3 = 4,\ 3^5 = 4{\cdot}3 = 5,\ 3^6 = 5{\cdot}3 = 1 \mod 7$$

Therefore
$$3^{100} = (3^6)^{16} \cdot 3^4 = 1^{16} \cdot 4 = 4 \mod 7.$$

16. A *triangulation* of an n-gon is a collection of $(n-3)$ non-intersecting diagonals. For $n = 4, 5, 6$:
a) Find the total number of triangulations for a regular n-gon
b) Describe the orbits and stabilizers of the action of D_n on the set of triangulations.

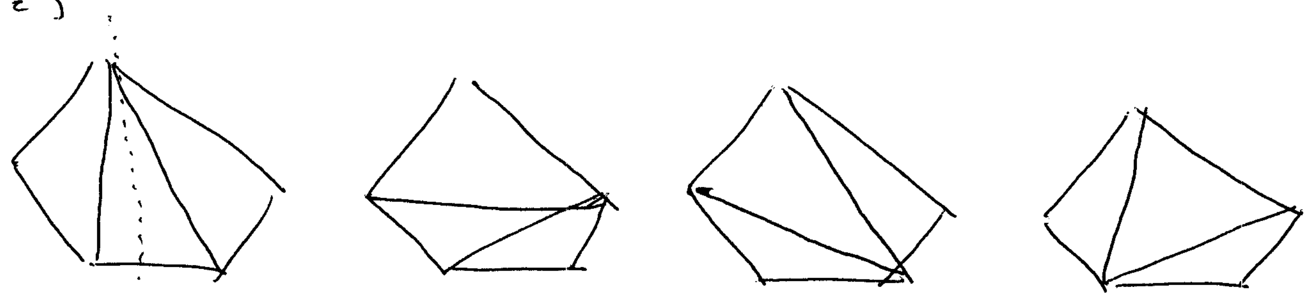
16 a) $n=4$

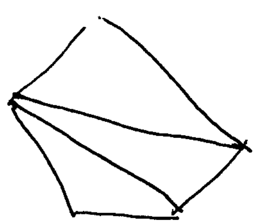


One orbit with 2 elements, Stabilizer $= \{e,$ reflections in diagonals, rotation by $180°\}$

$2 \cdot 4 = 8 = |D_4|$

b) $n=5$



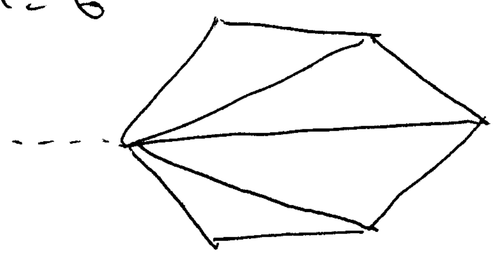


5 triangulations in a single orbit

Stabilizer $= \{e,$ reflection$\}$

$5 \cdot 2 = 10 = |D_5|$

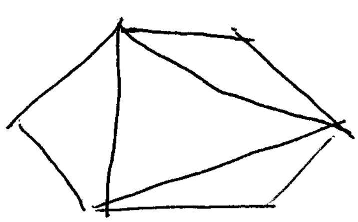
c) $n=6$



$\times 6$: $|\text{Orbit}| = 6$

Stabilizer $= \{e,$ reflection$\}$

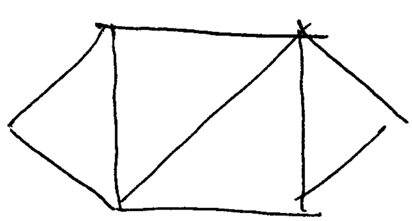
$6 \cdot 2 = 12 = |D_6|$



$\times 2$: $|\text{Orbit}| = 2$

Stabilizer $= D_3$

$2 \cdot 6 = 12 = |D_6|$



$\times 6$: $|\text{Orbit}| = 2$

Stabilizer $= \{e,$ rotation by $180°\}$

$2 \cdot 6 = 12 = |D_6|$

In total, 14 triangulations of a 6-gon