Page 34, ex. 5.1 (25 points) Write the following permutations as products of disjoint cycles:

\[(1 \ 2)(1 \ 3)(1 \ 4)(1 \ 5), \ (1 \ 2 \ 3)(2 \ 3 \ 4)(3 \ 4 \ 5), \ (1 \ 2 \ 3 \ 4)(2 \ 3 \ 4 \ 5), \ (1 \ 2)(2 \ 3)(3 \ 4)(4 \ 5)(5 \ 1)\].

Solution:

\[(1 \ 2)(1 \ 3)(1 \ 4)(1 \ 5) = (1 \ 5 \ 4 \ 3 \ 2), \ (1 \ 2 \ 3)(2 \ 3 \ 4)(3 \ 4 \ 5) = (1 \ 2)(4 \ 5), \]

\[(1 \ 2 \ 3)(2 \ 3 \ 4 \ 5) = (1 \ 2 \ 4 \ 5 \ 3), \ (1 \ 2)(2 \ 3)(3 \ 4)(4 \ 5)(5 \ 1) = (2 \ 3 \ 4 \ 5)\].

Page 69, ex. 2.3 (25 points) Let \(x, y, z\) and \(w\) be elements in a group \(G\).

a) Solve for \(y\) given that \(xyz^{-1}w = 1\).

b) Suppose that \(xyz = 1\). Does it follow that \(yzx = 1\)? Does it follow that \(yxz = 1\)?

Solution: (a) By multiplying the equation by \(x^{-1}\) on the left, we get \(yz^{-1}w = x^{-1}\).

By multiplying it by \(w^{-1}\) on the right, we get \(yz^{-1}w = x^{-1}w^{-1}\). Finally, by multiplying by \(z\) on the right, we get \(y = x^{-1}w^{-1}z\).

(b) If we solve the first equation for \(y\) similarly to (a), we get \(y = x^{-1}z^{-1}\), so \(yzx = x^{-1}z^{-1}zx = x^{-1}1x = 1\), so the second equation follows. On the other hand, the third equation does not follow from the first one: for \(x = (1 \ 2), y = (2 \ 3)\) and \(z = (1 \ 3 \ 2)\) we get

\(xyz = (1 \ 2)(2 \ 3)(1 \ 3 \ 2) = e, \) but \(yxz = (2 \ 3)(1 \ 2)(1 \ 3 \ 2) = (1 \ 2 \ 3)\).

Alternatively, one can pick any \(x, y\) such that \(xy \neq yx\) and \(z = (xy)^{-1}\).

A. (25 points) Find the number of even and odd permutations in \(S_n\) for all \(n\).

Solution: If \(f\) is an even permutation, define \(T(f) = f \cdot (1 \ 2)\). Since \(\text{sgn}(T(f)) = \text{sgn}(f) \cdot \text{sgn}(1 \ 2) = -\text{sgn}(f)\), permutations \(f\) and \(T(f)\) have opposite signs. Furthermore, \(T(T(f)) = f \cdot (1 \ 2) \cdot (1 \ 2) = f\), so the function \(T\) on the set of permutations is inverse to itself and hence a bijection.

Therefore \(T\) is a bijection from the set of even permutation to the set of odd permutations and these two sets have the same number of elements. Since the total number of permutations equals \(n!\), both the number of even permutations and the number of odd ones are equal to \(n!/2\).

B. (25 points) Present the following permutation as a product of non-intersecting cycles for all \(k\):

\[(1 \ 2 \ 3)^k = \underbrace{(1 \ 2 \ 3) \cdots (1 \ 2 \ 3)}_{k \text{ times}}\]

Solution: We have \((1 \ 2 \ 3)^2 = (1 \ 3 \ 2), \ (1 \ 2 \ 3)^3 = (1 \ 2 \ 3)(1 \ 3 \ 2) = e\). Therefore \(f^4 = f^3 \cdot f = e \cdot f = f, f^5 = f^3 \cdot f^2 = f^2, f^6 = f^3 \cdot f^3 = e\). In general, the powers of \(f\) keep repeating with period 3 and

\[f^k = \begin{cases} 
(1 \ 2 \ 3) & \text{if } k = 3n + 1, \\
(1 \ 3 \ 2) & \text{if } k = 3n + 2, \\
e & \text{if } k = 3n.
\]